

# Classification of Control Ensembles of Projective Points

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**Abstract**—We consider the problem of classification of unordered collections of complex projective points on the line (called ensembles) controlled by a scalar parameter with respect to projective transformations. The problem is reduced to that of classification of binary forms with control parameter with respect to the action of a certain pseudogroup. This problem is solved in two steps. First we consider the action of the pseudogroup on the infinite prolongation of the Euler differential equation and find the algebra of differential invariants of this action. Then, applying the technique of geometric theory of differential equations, we show that the triple of relations between the basis differential invariants and their invariant derivatives determines uniquely an equivalence class of controlled binary forms and, consequently, an equivalence class of controlled collections of projective points.

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## 1. INTRODUCTION

Consider an unordered collection of points on the complex projective line (we call such a collection an *ensemble*) controlled by a scalar parameter  $u$ . The action of the projective group on the projective line is lifted to its action on the set of all such ensembles.

The aim of the paper is to classify the ensembles with respect to the feedback transformations. The basic idea of the classification is the reduction of the problem to the classification of orbits of binary forms with control parameter with respect to the action of a certain pseudogroup  $G$ . Note that the classification of  $\mathrm{GL}_2(\mathbb{C})$ -orbits of binary forms of arbitrary degree was obtained in [1, 2]. Based on the ideas of [1, 2], we obtain a classification of controlled binary forms and, as a consequence, a classification of motions of controlled ensembles of projective points.

## 2. STATEMENT OF THE PROBLEM AND PRELIMINARIES

**2.1. Controlled ensembles.** Let  $\mathbb{C}P^1$  be a complex projective line. Consider an unordered collection  $\{P_1(u), \dots, P_n(u)\}$  of points on  $\mathbb{C}P^1$  depending on a parameter  $u \in \mathbb{C}$ .

**Definition 1.** Such a collection will be called a *controlled ensemble*.

It is natural to put the question on *classification* of controlled ensembles with respect to projective transformations of  $\mathbb{C}P^1$  and changes of the control parameter. This problem is a “controlled” analog of the classical geometric problem on classification of unordered collections of projective points, which marked the beginning of the classical theory of invariants [3].

As in the classical case, we reduce the problem of classification of controlled ensembles to the classification of “controlled binary forms”. For this, we consider the complex plane  $\mathbb{C}^2$  with coordinates  $(x, y)$  which generates the projective line  $\mathbb{C}P^1$  with homogeneous coordinates  $(x : y)$  in question. Then

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