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TWO-DIMENSIONAL HOMOLOGY OF THE COMPLEMENT OF AN ALGEBRAIC CURVE IN $\mathbb{C}^2$ AND $(\mathbb{C} \setminus \{0\})^2$

O.V. Yakovleva

1. Introduction

Let V be an irreducible algebraic curve in either the torus $T^2 = (\mathbb{C} \setminus \{0\})^2$, or the affine space $\mathbb{C}^2$, given by zeros of a Laurent polynomial

$$P(z) = \sum_{k \in K \subset \mathbb{Z}^2} C_k z^k$$

(in the case where the curve V is in $\mathbb{C}^2$, the polynomial $P(z)$ does not contain negative powers of z_1 and z_2). In the last century, in works of authors who traditionally studied curves in the projective compactification $\mathbb{CP}_2$ of the space $\mathbb{C}^2$, it was established that the genus of a smooth curve in $\mathbb{CP}_2$ depends only on the degree of the polynomial $P(z)$ (the Plücker formula: $\rho = \frac{(d-1)(d-2)}{2}$, where d is the total degree of $P(z)$). In that situation, the smoothness of a curve expresses the condition of general position. As was shown in [1], [2], it seems reasonable to treat the number $\rho = \frac{(d-1)(d-2)}{2}$ as the number of integer points in the interior of the triangle $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq d\}$, to which the projective space $\mathbb{CP}_2$ corresponds in the intrinsic way. The basic idea (see [1]) is that either a curve V (or a surface in T^n, or $\mathbb{C}^n$) which is not in general position in $\mathbb{CP}_2$ may be in general position in a certain other compactification of either T^2, or $\mathbb{C}^2$. It turned out that to this end toric compactifications associated with the Newton polyhedron Δ of the polynomial P are very convenient. In the conditions of general position (called in [1] Δ-nondegeneracy), the genus of V is expressed in terms of the number of interior points of the polyhedron Δ.

The aim of this article is to establish certain conditions of general position on the polynomial P, under which the dimensions of the homology groups $H_2(T^2 \setminus V)$ and $H_2(\mathbb{C}^2 \setminus V)$ can be expressed by means of a simple and effective formula.

We consider homology groups with coefficients in either $\mathbb{R}$, or $\mathbb{Z}$.

Let $\mathbb{X}$ be a smooth toric compactification of either T^2, or $\mathbb{C}^2$ (see [1], [3]). Recall that a toric space $\mathbb{X}$ associated with a complete fan Σ can be treated as the compactification $\mathbb{X} = T^2 \cup \sum T_j$ of the torus T^2 by the curves T_j "at infinity", which correspond bijectively to the generators $v_1, \ldots, v_d$ of the fan Σ. Each T_j is homeomorphic to the Riemann sphere $\overline{\mathbb{C}}$ (the projective line $\mathbb{CP}_1$), and $\sum T_j$ is the normal intersection in $\mathbb{X}$ (see [1]). In the case where the fan Σ contains the positive octant generated by the vectors $v_1 = (1, 0)$, $v_2 = (0, 1)$, the space

$$\mathbb{X} = T^2 \cup \sum_{j=1}^{d} T_j = \mathbb{C}^2 \cup \sum_{j=3}^{d} T_j$$

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