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A Nonlocal Problem for a First-Order Partial Differential Equation

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Abstract—In this paper we study a nonlocal problem for a first-order partial differential equation with an integral condition instead of the standard boundary one. We prove that the problem under consideration is uniquely solvable.

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1. THE PROBLEM

Consider the following partial differential equation:

$$Lu \equiv u_t + \operatorname{div}(A(x,t)\,u) + B(x,t)u = f(x,t) \tag{1}$$

on the cylinder $Q_T = \{(x,\tau) : x \in \Omega,\ 0 < \tau < T\}$, where $\Omega \subset \mathbb{R}^n$. For this equation we state the problem with the Cauchy initial condition

$$u(x,0) = \tau(x) \tag{2}$$

and the nonlocal one

$$u(x,t)|_{S_T} = \int_\Omega K(x,y,t)\,u(y,t)\,dy|_{S_T}, \tag{3}$$

where $\tau(x)$ and $K(x,y,t)$ are given. Here $A(x,t) = (A_1(x,t),\dots,A_n(x,t))$ is a vector-valued function.

We assume that the boundary $\Gamma = \partial\Omega$ is smooth and the surface $S_T = \{(x,t) : x \in \Gamma$ is smooth, $0 < t < T\}$ is the lateral surface of the cylinder Q_T.

Let the symbol $L_{2,1}(Q_T)$ denote the Banach space consisting of all functions $u(x,t)$ defined and Lebesgue measurable on Q_T with the finite norm

$$\|u\|_{2,1} = \int_0^T \left(\int_\Omega u^2(x,t)\,dx\right)^{1/2} dt.$$

We denote by S_t the lateral surface of Q_t and we do the scalar product as

$$(A(x,t),\,\mathbf{n}) = r(x,t),$$

where $\mathbf{n}$ is the outer normal vector. Assume that

$$r(x,t) \le 0,\ \ x \in \partial\Omega,\ \ t \in [0,T].$$

In what follows we denote by $V(Q_T)$ the Banach space consisting of all functions $u(x,t)$ defined and Lebesgue measurable on Q_T with the finite norm

$$\|u\|_{V(Q_T)} = \left(\sup_{t\in[0,T]} \int_\Omega u^2\,dx + \int_0^T \int_{\partial\Omega} r_1(x,t)\,u^2\,ds\,dt\right)^{1/2},$$

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