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ISOLATION: MOTIVATIONS AND APPLICATIONS

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Abstract

In this paper, we briefly review the origins of the isolation phenomenon and its applications. We discuss a stronger notion of double bubbles. We also show recent achievements in the study of lattice embeddings with the help of the isolation property.

Key words: Turing degrees, Ershov hierarchy, isolated degrees, lattice embeddings.

Introduction

An $n+1$-c.e. degree $\mathbf{d}$ is isolated by an n-c.e. degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{d}$ is the greatest n-c.e. degree below $\mathbf{d}$. The existence of such isolated $n+1$-c.e. degrees, for $n \geq 1$, can be obtained, in a nonuniform way, from Kaddah's thesis, where infima of n-c.e. degrees in different levels of the Ershov hierarchy are considered. The case when $n = 1$ was first proposed explicitly by Cooper and Yi in their paper [1], and the general case, i.e. when $n \geq 1$, was studied by LaForte in [2]. A recent construction of bubbles of Arslanov, Kalimullin and Lempp in [3] also gives rise to the existence of isolated degrees, and it is attempting to extend such bubble constructions to show that different (finite) levels in the Ershov hierarchy are not elementary equivalent.

In this paper, we briefly review the origins of the isolation phenomenon and how variants of this phenomenon can be applied to study local and global properties of the Ershov hierarchy. In Section 1, we first show how to obtain isolated degrees from Kaddah's thesis, and then give a brief description of early development in this area. In Section 2, we consider isolation from side, a property that was used by Yang and Yu to show that the c.e. degrees is not a Σ_1 substructure of d.c.e. degrees. This phenomenon has been extended to n-c.e. degrees by Cai, Shore and Slaman [4]. We are concerned with those nonisolated degrees that can be isolated from side, which are nontrivial extensions of Cooper and Yi's isolated degrees. In Section 3, we give a direct construction of a bubble, a work of Arslanov, Kalimullin and Lempp. That is, we will provide a construction of a d.c.e. degree $\mathbf{d}$ and a c.e. degree $\mathbf{a} < \mathbf{d}$ such that every d.c.e. degree $\mathbf{e}$ below $\mathbf{d}$ is comparable with $\mathbf{a}$. Obviously, $\mathbf{d}$ is isolated by $\mathbf{a}$. We believe that, like the isolation phenomenon, the bubble phenomenon is an important tool for studying the structural properties of the Ershov hierarchy. In Section 4, we show recent development of applications of isolation to lattice embeddings. This project was initiated by Wu in his thesis [5] and has come to a highlight in a recent work of Fang, Liu and Wu, who proved in [6] that any nonzero cappable c.e. degree can have a d.c.e. degree with almost universal cupping property as its complement.

Our notation and terminology are standard and generally follow Soare [7]. We suggest the readers to refer Cooper's paper [8] and Arslanov's paper [9] for the general idea on local degree theory.

1. Kaddah's work and isolation

A Turing degree is properly d.c.e. if it contains a d.c.e. set, but no c.e. set. Cooper proved in [10] the existence of properly d.c.e. degrees, and Lachlan observed that any nonzero d.c.e. degree bounds a nonzero c.e. degree. That is, given a d.c.e. set D with an effective approximation $\{D_s : s \in \omega\}$, the associated set

$$L(D) = \{\langle x, s\rangle : x \in D_s - D\}$$

is c.e., and is Turing reducible to D, while D is c.e. in $L(D)$. If D is c.e. and $\{D_s : s \in \omega\}$ is an effective enumeration of D, then $L(D)$ is empty. On the other hand, if D has proper d.c.e. degree, then $L(D)$ is not computable. *$L(D)$ is called the Lachlan set of D with reference to the enumeration $\{D_s : s \in \omega\}$.* Lachlan's observation shows that the d.c.e. degrees are downwards dense, which is also true for the c.e. degrees. However, the d.c.e. degrees are not dense, and hence these two degree structures are not elementarily equivalent.

Theorem 1 (Nondensity Theorem for $\mathcal{D}_2$ [11]). *There exists a maximal d.c.e. degree $\mathbf{d} < \mathbf{0}'$, and hence the d.c.e. degrees are not dense.*

The fact that these structures are not elementarily equivalent was first proved by Arslanov [12] and Downey [13], who proved that any nonzero d.c.e. degree is cuppable, and that the diamond lattice can be embedded into the d.c.e. degrees preserving 0 and 1, respectively.

In contrast to this nondensity theorem, Ishmukhametov [14] and, independently, Cooper and Yi [1] proved that any nonempty interval $[\mathbf{a}, \mathbf{d}]$, with $\mathbf{a}$ c.e., contains infinitely many d.c.e. degrees, a weak density theorem of d.c.e. degrees.

Theorem 2 [1, 14]. *If $\mathbf{d}$ is a d.c.e. degree and $\mathbf{a} < \mathbf{d}$ is a c.e. degree, then there is a d.c.e. degree $\mathbf{c}$ between $\mathbf{a}$ and $\mathbf{d}$.*

Here we cannot require the degree $\mathbf{c}$ above be c.e., as there are a c.e. degree $\mathbf{a}$ and a d.c.e. degree $\mathbf{d} > \mathbf{a}$ such that no c.e. degree is between $\mathbf{a}$ and $\mathbf{d}$. This can be obtained from the following theorem of Kaddah.

Theorem 3 [15]. *Every low c.e. degree is branching in the d.c.e. degrees.*

Let $\mathbf{a}$ be a low, nonbranching, c.e. degree, and let $\mathbf{d}, \mathbf{e}$ be two d.c.e. degrees above $\mathbf{a}$ such that $\mathbf{a}$ is the infimum of $\mathbf{d}$ and $\mathbf{e}$ in the d.c.e. degrees. Then one of the intervals $(\mathbf{a}, \mathbf{d})$, $(\mathbf{a}, \mathbf{e})$ contains no c.e. degrees, as $\mathbf{a}$ is assumed to be nonbranching.

Cooper and Yi first noticed this structural phenomenon and proposed the notion of isolation explicitly in their paper [1].

Definition 1 [1]. A d.c.e. degree $\mathbf{d}$ is isolated by a c.e. degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{d}$ is the greatest c.e. degree below $\mathbf{d}$. A d.c.e. degree $\mathbf{d}$ is isolated if it is isolated by some c.e. degree $\mathbf{a}$. A d.c.e. degree is nonisolated if it is not isolated.

After showing the existence of the isolated degrees, Cooper and Yi continued to show the existence of the nonisolated degrees, where a d.c.e. degree $\mathbf{d}$ is nonisolated if no c.e. degree below $\mathbf{d}$ can isolate $\mathbf{d}$. Cooper and Yi actually proved the existence of a properly d.c.e. degree as a minimal upper bound of a uniformly c.e. sequence of degrees, an even stronger result. These two kinds of degrees are proved to be dense in the c.e. degrees.

Theorem 4 [16–18]. *Both the isolated d.c.e. degrees and the nonisolated d.c.e. degrees are dense in the c.e. degrees.*

Theorem 4 says that the isolated degrees could be as close to the isolating degrees as wanted. Ishmukhametov and Wu proved that in terms of the high/low hierarchy, the isolated d.c.e. degree and the isolating degree can be quite far from each other.

Theorem 5 [19, 20]. *There is a high d.c.e. degree* $\mathbf{d}$ *isolated by a low c.e. degree* $\mathbf{c}$. *Such a c.e. degree* $\mathbf{c}$ *can be found below any nonzero c.e. degree* $\mathbf{a}$.

Cooper [21] proved in 1974 that any high c.e. degree bounds a minimal pair, and hence no high c.e. can be nonbounding. However, there do exist high d.c.e. nonbounding degrees, as first constructed by Chong, Li and Yang in [22] by a fairly complicated $0'''$ argument. Theorem 5 can provide another proof of this result, as if we first take $\mathbf{a}$ as a nonbounding degree, and then apply Theorem 5 to obtain $\mathbf{c}$ and $\mathbf{d}$. Obviously, $\mathbf{c}$ is also nonbounding, which implies that $\mathbf{d}$ is also nonbounding (and also high).

In [18], Arslanov, Lempp and Shore showed the existence of the nonisolating degrees, and proved that these degrees are downwards dense in the c.e. degrees, and can occur in every jump class. In contrast to this, Cooper, Salts and Wu proved in [23] that the nonisolating degrees are upwards dense in the c.e. degrees. Furthermore, Salts proved in [24] that the nonisolating degrees are not dense in the c.e. degrees.

Theorem 6 [24]. *There is an interval of c.e. degrees,* $[\mathbf{a}, \mathbf{c}]$, *each of which isolates a d.c.e. degree.*

Recent work of Wu and Yamaleev[1] shows that such an interval can be large. That is, $\mathbf{c}$ above can be high and $\mathbf{a}$ can be low.

Lachlan [25] proved in 1966 that the infimum of two c.e. degrees in the c.e. degrees and the infimum of two c.e. degrees in the Δ^0_2 degrees coincide. In contrast to this, in [15], Kaddah proved that the infima of n-c.e. degrees in the n-c.e. degrees can be different from that of these two n-c.e. degrees in the $(n+1)$-c.e. degrees.

Theorem 7 [15]. *For each* $n \geq 2$, *there are* n*-c.e. degrees* $\mathbf{d}, \mathbf{e}$ *such that they have* $\mathbf{f}$ *as infimum in the* n*-c.e. degrees, and there is an* $(n+1)$*-c.e. degree* $\mathbf{x}$ *with* $\mathbf{f} < \mathbf{x} < \mathbf{d}, \mathbf{e}$.

This implies isolation at higher levels in the Ershov hierarchy, as $\mathbf{x}$ shows that $\mathbf{d}$ and $\mathbf{e}$ do not have $\mathbf{f}$ as their infimum in the $(n+1)$-c.e. degrees. Following Cooper and Yi, we can say that $\mathbf{x}$ is isolated by $\mathbf{f}$ in the n-c.e. degrees. Liu, Wang and Wu[2] proved that such isolation pairs, $\mathbf{x}$ and $\mathbf{f}$, are dense in the c.e. degrees. This isolation result was proved previously by LaForte in [2] by a different approach.

2. Variants of isolation

Arslanov's cupping theorem shows that the structures of the c.e degrees and the d.c.e degrees differ at Σ_3 level, and Cooper et al.'s proof of the existence of incomplete maximal d.c.e. degrees, and Downey's diamond theorem show that these two structures differ at Σ_2 level. It becomes interesting to consider whether two structures differ at Σ_1 level.

Say that nonzero c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a *Slaman triple* if $\mathbf{b} \not\geq \mathbf{c}$, and for any nonzero c.e. degree $\mathbf{w} \leq \mathbf{a}$, $\mathbf{w} \vee \mathbf{b} \geq \mathbf{c}$. It is easy to check that $\mathbf{a}$ and $\mathbf{b}$ above form a minimal pair, and Shore and Slaman proved in 1993 that every high c.e. degree bounds a Slaman triple.

In 1983, Slaman proved the following strengthened version of Slaman triples.

[1] A large interval of isolating degrees, in preparation.

[2] An alternative approach of isolated $(n+1)$-c.e. degrees, in preparation.

Theorem 8 (Slaman 1983). *There are c.e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and a* Δ_2^0 *degree* $\mathbf{d}$ *with* $\mathbf{0} < \mathbf{d} < \mathbf{a}$ *such that* (1) $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *form a Slaman triple,* (2) $\mathbf{d} \vee \mathbf{b} \not\geq \mathbf{c}$.

Theorem 8 says that no nonzero c.e. degree $\mathbf{w}$ below $\mathbf{a}$ has the property that $\mathbf{c} \not\leq \mathbf{w} \vee \mathbf{b}$, while there is a nonzero Δ_2^0 degree $\mathbf{d}$ below $\mathbf{a}$ that has this property. This is a Σ_1 property, which provides a Σ_1 difference between the c.e. degrees and the Δ_2^0 degrees.

In [26], Yang and Yu proved that the c.e. degrees and the d.c.e. degrees also differ at Σ_1 level, by modifying this proof of Slaman, where another parameter is introduced to handle the degrees of Lachlan's sets of $\mathbf{d}$, if $\mathbf{d}$ is d.c.e.

Theorem 9 [26]. *There are c.e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}$ *and a d.c.e. degree* $\mathbf{d} < \mathbf{a}$ *such that* $\mathbf{d} \vee \mathbf{b} \not\geq \mathbf{c}$, $\mathbf{d} \not\leq \mathbf{e}$, *and for any c.e. degree* $\mathbf{w} < \mathbf{a}$, *either* $\mathbf{w} \vee \mathbf{b} \geq \mathbf{c}$ *or* $\mathbf{w} \leq \mathbf{e}$.

This proof was recently extended by Cai, Shore and Slaman to prove that for any $m < n$, the m-c.e. degrees is not a Σ_1-substructure of the n-c.e. degrees.

Theorem 10 [4]. *There are c.e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}$ *and an* $(n+1)$*-c.e. degree* $\mathbf{d} < \mathbf{a}$ *such that* $\mathbf{d} \vee \mathbf{b} \not\geq \mathbf{c}$, $\mathbf{d} \not\leq \mathbf{e}$, *and for any* n*-c.e. degree* $\mathbf{v} < \mathbf{a}$, *either* $\mathbf{v} \vee \mathbf{b} \geq \mathbf{c}$ *or* $\mathbf{v} \leq \mathbf{e}$.

These two results give rise to another isolation. That is, all the n-c.e. degrees below $\mathbf{d}$ are isolated, or bounded, by c.e. degree $\mathbf{e}$. Note that we have two cases: either there is such an $\mathbf{e}$ below $\mathbf{d}$ ($\mathbf{d}$ is isolated by $\mathbf{e}$ in the n-c.e. degrees) or $\mathbf{e}$ cannot be below $\mathbf{d}$ ($\mathbf{d}$ is nonisolated). We consider the case when $n = 1$, and in the latter case, we say that d.c.e. degree $\mathbf{d}$ is isolated *from side nontrivially*: $\mathbf{d}$ itself is nonisolated, and there is a c.e. degree $\mathbf{e}$ incomparable with $\mathbf{d}$, bounding all the c.e. degrees below $\mathbf{d}$.

We comment here that in the results mentioned above, it is hard to insert $\mathbf{e}$ below $\mathbf{d}$. That is, it is hard to make $\mathbf{d}$ isolated. For Cai, Shore and Slaman's result, for $n \geq 2$, if we really want to move $\mathbf{e}$ below $\mathbf{d}$, then $\mathbf{e}$ cannot be c.e. anymore, and we will make it n-c.e. (this is what we want).

In the following, we show how to construct a d.c.e. degree isolated from side nontrivially. We will build d.c.e. sets B, D and a c.e. set C satisfying the following requirements:

$$\mathcal{G} : B \leq_T C,$$

$$\mathcal{P}_e^D : \ D \neq \Phi_e^C,$$

$$\mathcal{P}_e^B : \ B \neq \Phi_e^{W_e} \ \vee \ W_e \neq \Psi_e^B,$$

$$\mathcal{Q}_e : \ \Phi_e^B = W_e \Rightarrow (\exists \text{ c.e. } U_e \leq_{\mathrm{T}} B)(\forall i)(U_e \neq \Psi_i^{W_e}),$$

$$\mathcal{R}_e : \ \Phi_e^{B \oplus D} = W_e \Rightarrow \exists \Gamma_e(\Gamma_e^B = W_e),$$

where $\{\langle \Phi_e, \Psi_e, W_e \rangle : e \in \omega\}$ is a standard enumeration of all $\langle \Phi, \Psi, W \rangle$ for which Φ, Ψ are partial computable functionals and W is a c.e. set.

Let $\mathbf{b}, \mathbf{c}, \mathbf{d}$ be the Turing degrees of B, C and $B \oplus D$, respectively. By the $\mathcal{Q}$-requirement, $\mathbf{b} \leq \mathbf{c}$. All the $\mathcal{P}^B$-requirements ensure that $\mathbf{b}$ is a properly d.c.e. degree, and hence $\mathbf{b} < \mathbf{c}$. By the $\mathcal{Q}$-strategies, $\mathbf{b}$ is nonisolated. The $\mathcal{R}$-strategies ensure that all the c.e. degrees below $\mathbf{d}$ are also below $\mathbf{b}$, hence below $\mathbf{c}$. According to Wu [5], $\mathbf{d}$ is pseudo-isolated.

For the $\mathcal{G}$-requirement, we construct a p.c. functional Λ such that $B = \Lambda^C$, and for a number x, if we put a number x into B, or extract it from B, at a stage s, we always put $\lambda(x)[s]$ into C automatically. Obviously, Λ is totally defined.

A $\mathcal{P}_e^D$-strategy is a standard Friedberg–Muchnik strategy, and a $\mathcal{P}_e^B$-strategy is a strategy used in the construction of proper d.c.e. degrees. An $\mathcal{R}_e$-strategy is similar to an isolation strategy. That is, we check at every expansionary stage whether Γ_e^B and W_e agree, and if not, suppose they differ at x, then we extract relevant numbers out of D to recover a computation $\Phi_e^{B\oplus D}(x)$ to a previous one, which has value 0. This creates a disagreement between $\Phi_e^{B\oplus D}$ and W_e, and the requirement is satisfied.

A $\mathcal{Q}_e$-strategy, ζ say, attempts to construct a c.e. set U_e such that if $W_e = \Phi_e^B$, then $U_e \leq_T B$ and for all $i \in \omega$, $U_e \neq \Psi_i^{W_e}$. Define ζ-expansionary stages in a standard way, and ζ has two outcomes: 0 and 1, where 0 stands for the case that there are infinitely many ζ-expansionary stages, and 1 for the other case.

Suppose that ζ has outcome 0. The construction of U_e will be carried out by $\mathcal{Q}_e$'s substrategies, $\mathcal{S}_{e,i}$, $i \in \omega$, which are arranged in the cone below $\zeta^\frown\langle 0\rangle$:

$$\mathcal{S}_{e,i} : U_e \neq \Psi_i^{W_e}.$$

Let β be an $\mathcal{S}_{e,i}$-strategy. Then β tries to figure out a disagreement between U_e and $\Psi_i^{W_e}$ or between W_e and Φ_e^B.

(1) Choose x as a fresh number.

(2) Wait for a stage s such that

$$\Psi_{i,s}^{W_{e,s}}(x)\downarrow = 0 \quad \text{and } W_{e,s} \upharpoonright \psi_{i,s}(x) = \Phi_{e,s}^{B_s} \upharpoonright \psi_{i,s}(x).$$

(*If this never happens, then x is a witness to the success of $\mathcal{S}_{e,i}$.*)

(3) Put x into U_e and B. Protect $B \upharpoonright s$ from other strategies.

(4) Wait for a stage s' such that

$$\Psi_{i,s'}^{W_{e,s'}}(x)\downarrow = 1 \quad \text{and } W_{e,s'} \upharpoonright \psi_{i,s}(x) = \Phi_{e,s'}^{B_{s'}} \upharpoonright \psi_{i,s}(x).$$

(*If this never happens, then again x is a witness to the success of $\mathcal{S}_{e,i}$. If it happens, then the change in $\Psi_i^{W_e}(x)$ between stages s and s' can only be brought about by a change in $W_e \upharpoonright \psi_{i,s}(x)$, which is irreversible since W_e is a c.e. set.*)

(5) Remove x from B and protect $B \upharpoonright s$ from other strategies.

(*Now x is a permanent witness to the success of $\mathcal{S}_{e,i}$ because*

$$\Phi_e^B \upharpoonright \psi_{i,s}(x) = \Phi_{e,s}^B \upharpoonright \psi_{i,s}(x) = W_{e,s} \upharpoonright \psi_{i,s}(x) \neq W_e \upharpoonright \psi_{i,s}(x).$$

That is, taking x from B leads to a global win on $\mathcal{Q}_e$, and U_e is no longer needed, so we don't need to care about the loss of B-permission for x (which is left in U_e).)

In the construction, since we are constructing B d.c.e., the $\mathcal{R}$-strategy is a little more complicated than the standard isolation strategy. Suppose that $\Gamma_\eta^B(x)$ gets defined at stage s_0, and a $\mathcal{P}^B$ (or an $\mathcal{S}$)-strategy ξ with lower priority enumerates a number $z < \varphi_\eta(x)$ into B at a stage $s_1 > s_0$. This enumeration forces (and allows) us to lift $\gamma_\eta(x)$ to a larger number, $\gamma_\eta(x)[s_1]$. Later, at stage s_2, to get a disagreement, ξ, or its mother node, takes z out, and thus $\gamma_\eta(x)$ returns to its definition at stage s_0. Such a variation does no harm to the disagreement strategy of $\mathcal{R}$-strategy, η say. Suppose that η observes at some stage between s_1 and s_2 that $\Gamma_\eta^B(x)$ is incorrect and performs the disagreement strategy; then ξ will be initialized, and so ξ has no chance to take z out. In this case, s_2 does not exist. In case that s_2 does exist, and η finds an incorrectness of $\Gamma_\eta^B(x)$ at stage $s_3 > s_2$, since at stage s_2, $\gamma_\eta(x)$ returns to that of stage s_0, we have

$$B_{s_2} \upharpoonright \varphi_{e,s_0} = B_{s_0} \upharpoonright \varphi_{e,s_0}.$$

Now by the fact that $\gamma_\eta(x)[s_3] = s_0$, we have

$$B_{s_3} \upharpoonright \varphi_{e,s_0} = B_{s_0} \upharpoonright \varphi_{e,s_0}.$$

This guarantees the success of η's disagreement strategy.

$\mathbf{d}$ above is called a pseudo-isolated degree, as a d.c.e. degree $\mathbf{b} < \mathbf{d}$ bounds all the c.e. degrees below $\mathbf{d}$. Note that $\mathbf{d}$ is nonisolated, as if $\mathbf{a}$ is a c.e. degree below $\mathbf{d}$, then $\mathbf{a}$ is also below $\mathbf{b}$. As $\mathbf{b}$ is nonisolated, there is a c.e. degree $\mathbf{e}$ below $\mathbf{b}$ (hence below $\mathbf{d}$) but not below $\mathbf{a}$. Wu proved in [27] that the pseudo-isolated d.c.e. degrees are dense in the c.e. degrees.

3. Double bubbles: a stronger notion

In this section, we consider a phenomenon called bubbles, which was discovered by Arslanov, Kalimullin and Lempp in their work [3]. A basic fact about isolation is that $\mathbf{a}$ isolates $\mathbf{d}$ if and only if $\mathbf{a}$ and $\mathbf{d}$ have the same lower cones in the c.e. degrees. Extending this concept, we consider the case when all the d.c.e. degrees below $\mathbf{d}$ are comparable with $\mathbf{a}$.

Fix a d.c.e. degree $\mathbf{d}$. Let $L(\mathbf{d})$ be the collection of all Lachlan sets $L(D)$, where D is a d.c.e. set in $\mathbf{d}$. It's easy to see that $\mathbf{d}$ is isolated by $\mathbf{a}$ if and only if each $X \in L(\mathbf{d})$ has its degree $\deg(X) \leq \mathbf{a}$. In [28], Ishmukhametov proved that there exist a c.e. degree $\mathbf{a}$ and a d.c.e. degree $\mathbf{d}$ such that $L(\mathbf{d}) \subseteq \mathbf{a}$, and called such degrees $\mathbf{d}$ *exact* degrees. Obviously, all exact degrees are isolated by the degree of Lachlan sets. Ishmukhametov also proved in [28] that there exist isolated non-exact degrees.

Say that two nonzero d.c.e. degrees $\mathbf{d}$ and $\mathbf{a}$ (together with $\mathbf{0}$) form a bubble if all the d.c.e. degrees below $\mathbf{d}$ are comparable with $\mathbf{a}$. Obviously, any d.c.e. degree in the interval $(\mathbf{a}, \mathbf{d})$ and $\mathbf{a}$ also form a bubble. Arslanov, Kalimullin and Lempp in their recent work [3] proved the existence of such a bubble and that the degree $\mathbf{a}$ in any bubble must be c.e. The construction has special difficulties and it is still unknown whether such a structural phenomenon can be combined with other properties (in a similar way like isolated degrees, see, e.g., recent work of Wu [29] and his joint works with Fang and Liu [6] and [30]).

Arslanov, Kalimullin and Lempp also proved in [3] the following important theorem.

Theorem 11 [3]. *Let D and A be d.c.e. sets with $D \not\leq_T A$, and X be a c.e. set such that $X \leq_T D, A \not\leq_T X$, and both D and A are c.e. in X. Then there exists a d.c.e. set U with $X \leq_T U \leq_T D$, and U and A are Turing incomparable.*

Theorem 11 implies that for any bubble pair $\mathbf{a} < \mathbf{d}$, the c.e. degrees $\mathbf{x}$ such that $\mathbf{d}$ is relatively enumerable in and above $\mathbf{x}$ should be above or equal to $\mathbf{a}$. However, by Sacks splitting theorem (avoiding the upper cone of $\mathbf{a}$), we have that $\mathbf{x}$ cannot be strictly above $\mathbf{a}$. That is, $\mathbf{x}$ and $\mathbf{a}$ are the same, and $\mathbf{d}$ is an exact degree, and hence isolated. The proof of the existence of exact d.c.e. degrees is simpler than the one for bubbles.

One property of bubbles is that the splittings of the top d.c.e. degree of a bubble $\mathbf{a} < \mathbf{d}$ are always above $\mathbf{a}$. A related topic, nonsplittability avoiding upper cones, was first proposed by Cooper and Li in [31].

Definition 2. Given d.c.e degrees $\mathbf{a} < \mathbf{d}$,

(1) a splitting $\mathbf{x}_0$, $\mathbf{x}_1$ of $\mathbf{d}$ is *nontrivial* if $\mathbf{d} \not\leq \mathbf{x}_i$ for $i = 0, 1$;

(2) $\mathbf{d}$ *is splittable above* $\mathbf{a}$ if there exist a nontrivial splitting of $\mathbf{d} = \mathbf{x}_0 \cup \mathbf{x}_1$ such that $\mathbf{a} \leq \mathbf{x}_i$ for $i = 0, 1$;

(3) $\mathbf{d}$ *is splittable avoiding the upper cone* of $\mathbf{a}$ if there exists a nontrivial splitting of $\mathbf{d} = \mathbf{x}_0 \cup \mathbf{x}_1$ such that $\mathbf{a} \not\leq \mathbf{x}_i$ for $i = 0, 1$.

Note that if $\mathbf{d}$ is nonsplittable avoiding the upper cone of $\mathbf{a}$, then $\mathbf{d}$ is also nonsplittable avoiding the upper cone of any degree $\mathbf{e}$ below $\mathbf{a}$. On the other hand, in [32], Yamaleev proved that if $\mathbf{a}$ and $\mathbf{d}$ are properly d.c.e. degrees and there is no c.e. degree between them, then $\mathbf{d}$ can always be splittable avoiding the upper cone of $\mathbf{a}$.

This result is interesting due to the following reason. Assume that both $\mathbf{a} < \mathbf{d}$ are d.c.e. degrees and each d.c.e. degree in the interval $(\mathbf{a}, \mathbf{d}]$ is nonsplittable avoiding the upper cone of $\mathbf{a}$; then we call this interval a *nonsplitting interval*. (It is easy to see that if $\mathbf{a} < \mathbf{d}$ form a bubble, then $(\mathbf{a}, \mathbf{d}]$ is a nonsplitting interval.) Sacks' splitting theorem (avoiding upper cones) implies that no c.e. degree is in this interval. By Yamaleev's result mentioned above, $\mathbf{a}$ is c.e., and hence $\mathbf{d}$ is isolated by $\mathbf{a}$.

Below we provide a sketch of the proof of a bubble, which contains all features of constructions of d.c.e. degrees which are nonsplittable avoiding upper cones and also constructions of nonsplitting intervals.

Theorem 12 [3]. *There exist a c.e. degree* $\mathbf{a}$ *and a d.c.e. degree* $\mathbf{d}$ *such that* $\mathbf{0} < \mathbf{a} < \mathbf{d}$ *and any d.c.e. degree* $\mathbf{u} \leq \mathbf{d}$ *is comparable with* $\mathbf{a}$.

Sketch of proof. In the sketch, we will provide the basic idea of constructing c.e. set A and d.c.e. set D, including individual strategies and the interactions between these strategies. A and D are constructed to meet the following requirements:

$$\mathcal{P}_e : A \neq \Psi_e,$$

$$\mathcal{S}_e : D \neq \Theta_e^A,$$

$$\mathcal{R}_e : U_e = \Phi_e^{A \oplus D} \rightarrow (U_e = \Gamma_e^A \vee A = \Delta_e^U),$$

where $\{\langle \Phi_e, \Psi_e, \Theta_e \rangle\}_{e \in \omega}$ is an effective enumeration of all p.c. functionals, and $\{U_e\}_{e \in \omega}$ is an effective enumeration of all d.c.e. sets. Obviously, if A and D satisfy these requirements, then the degrees of A and $A \oplus D$ form a bubble, as wanted. Later we will often omit indices.

A $\mathcal{P}$-requirement and an $\mathcal{S}$-requirement can be satisfied by the Friedberg – Muchnik strategy (or a variant of it). For an $\mathcal{R}$-strategy, we assume that there are infinitely many expansionary stages (approximating $U = \Phi^{A \oplus D}$), and we try first to build a p.c. functional Γ at these stages. It can happen that some $\mathcal{S}$-strategy below it enumerates a number x into D, and this enumeration can change computation $\Phi^{A \oplus D}(y)$, hence allowing U to change at y. We could not change $\Gamma^A(y)$ since it requires changes of A on small numbers. *In the construction of isolated degrees, we just need to extract x from D, making $U(y) \neq \Phi^{A \oplus D}(y)$.* In our construction, if $U(y)$ changes from 1 to 0, i.e. y leaves U, then we act in the same way: extract x from D, making $U(y) = 0 \neq 1 = \Phi^{A \oplus D}(y)$, and hence satisfy this $\mathcal{R}$-requirement. However, if $U(y)$ changes from 0 to 1, i.e. y enters U, we could not just simply extract x out of D, as $U(y)$ can change from 1 to 0 later, and our effort on diagonalization fails. So if we see that $U(y)$ changes from 0 to 1, instead of making diagonalization immediately, we will turn to extend the definition of Δ^U on more arguments, z say, with y less than the δ-use. If later we want to enumerate z into A, we will need to force y out of U to undefine $\Delta^U(z)$, and we make it by extracting x from D to recover $\Phi^{A \oplus D}(y) = 0$, and now either y keeps in U (we get $U(y) = 1 \neq 0 = \Phi^{A \oplus D}(y)$) or y leaves U and we have $\Delta^U(z)$ undefined, and we can now enumerate z into A.

An $\mathcal{R}$-strategy has three outcomes Δ, Γ, fin with order $\Delta <_L \Gamma <_L fin$. If there are only finitely many expansionary stages, then fin is the correct outcome. Otherwise, there are infinitely many expansionary stages, and if from some stage on, Γ^A keeps correct in the remainder of the construction, then Γ is the correct outcome. If each version of Γ^A appears incorrect, then Δ^U will be defined infinitely many times, and Δ is the correct outcome. Note that whenever Δ-outcome appears to be correct, the current version of Γ^A becomes invalid, and we will start a new version of Γ^A.

For convenience, we use x as witnesses for $\mathcal{S}$-strategies (can be enumerated into D and perhaps removed out later), z as witnesses for $\mathcal{P}$-strategies (can be enumerated into A) and y for elements of U (can be in or not in U).

As described before, if an $\mathcal{S}$-strategy, α say, is working below the outcome Γ of an $\mathcal{R}$-strategy, τ say, then α works in a standard Friedberg–Muchnik manner, trying to find a witness x to satisfy the requirement. If the enumeration of x causes $U(y)$ different from $\Gamma^A(y)$, without loss of generality, we assume that $U(y)$ changes from 0 to 1, then τ will have outcome Δ, i.e. τ tries to extend Δ^U on more arguments.

A $\mathcal{P}$-strategy, β say, below the outcome Δ of an $\mathcal{R}$-strategy τ, tries to enumerate its witness z into A. β cannot enumerate z into A immediately if $\Delta^U(z)$ is defined, as 0, at the moment. Here is the point: when β chooses z as its witness, τ has outcome Δ, which is caused by an enumeration of an $\mathcal{S}$-strategy, say α (below outcome Γ), at a stage s, which makes $U(y)$ to change from 0 to 1. Obviously, when z is selected, z is selected much bigger, and especially $z > s$. Now if β wants to enumerate z into A, then the action is to extract x out of D and also enumerate z into A simultaneously. Of course, we can enumerate z later, after we see that $U(y)$ changes back to 0. We prefer to enumerate z into A at the same time when we extract x out of D, as $z > s$, which is bigger than the use in $\Phi^{A\oplus D}(y)[s]$. If $U(y)$ does not change back to 0, then we win as $\Phi^{A\oplus D}(x) = 0 \neq 1 = U(y)$, and the $\mathcal{R}$-requirement is satisfied. Otherwise, $\Delta^U(z)$ (and also $\Delta^U(s)$) is undefined, which makes β's enumerations into A consistent.

As U is assumed to be d.c.e., after y leaves U, it can never come back. Due to this, once β enumerates z into A, z remains in A, as $\Delta^U(z)$ will be redefined as 1 later and forever. The situation is quite different for the case when U is 3-c.e., as described in [3].

We now consider more complicated interactions among several strategies.

- **$\mathcal{P}$ below Δ-outcome of $\mathcal{R}_2$ below Γ-outcome of $\mathcal{R}_1$.**

A generic case is that after we put a number x_2 into D, Γ_1^A is defined at some point y^1, and extracting x_2 from D may now change $U_1(y^1)$, and the action described above to recover a computation does not apply here, as we are making D d.c.e. The idea here is that whenever we extract a number x_2 from D, besides enumerating z into A, we also enumerate into A a number s, the stage at which x_2 is enumerated into D. Enumerating z into A is for the sake of the $\mathcal{P}$-strategy, and enumerating s into A is to undefine $\Gamma_1^A(y)$, which are defined after stage s. This idea is exactly the same as that in the construction of isolated degrees, to maintain the consistency between $\mathcal{R}$-strategies. We use $s(x)$ to denote the stage at which x is enumerated into D. It is a routine to show that for a particular n, $\Gamma_1^A(n)$ can be undefined in this way by at most finitely many times, which ensures that if $\Phi_1^{A\oplus D}(n)$ converges, then $\Gamma_1^A(n)$ is defined.

- **$\mathcal{P}$ below Δ-outcomes of $\mathcal{R}_2$ and $\mathcal{R}_1$.**

For simplicity, we use Δ_1 and Δ_2 to denote the Δ-outcomes of $\mathcal{R}$-strategies τ_1 and τ_2, respectively, where τ_2 is below outcome Δ_1. Let β be a $\mathcal{P}$-strategy below τ_2's outcome Δ_2. We now describe how β works below these two Δ-outcomes.

Recall that for any $\mathcal{R}$-strategy, when it turns to have outcome Δ, it is caused by an enumeration of some $\mathcal{S}$-strategy below outcome Γ. Here is the idea: *when an $\mathcal{S}$-strategy α_2 below τ_2's outcome Γ_2 sees that $\Phi_{i_2}^A(x_2)$ converges to 0, at stage s_1 say, instead of enumerating x_2 into D immediately, it waits for the next time when α_2 is visited again, which will actually show that an $\mathcal{S}$-strategy, α_1 say, below τ_1's outcome Γ_1, already enumerates a witness x_1, being selected after stage s_1.* Note that at this stage, $s(x_1)$ is bigger than the uses of all computations seeing at stage s_1. Now assume that α_2 is visited again at stage s_2, then at this stage, α_2 enumerates x_2 into D, so $s(x_2) = s_2$. Without loss of generality, suppose that this enumeration leads τ_2 to have outcome Δ_2, and β selects a number z as its witness. We then associate z with x_1 and x_2, which means that enumerating z into A and extracting x_1 and x_2 out of D should happen at the same time. Thus, we have $x_2 < x_1 < s(x_1) < s(x_2) < z$. Assume later that β wants to enumerate z into A; it will do so and at the same time extract both x_2, x_1 from D and enumerate $s(x_1), s(x_2)$ into A. As discussed before, if we have a new τ_1-expansionary stage, then U_1 should have a change on the associated number, y_1 say, which undefines $\Delta_1^{U_1}(s(x_1))$, $\Delta_1^{U_1}(s(x_2))$ and $\Delta_1^{U_1}(z)$. Also, if we have a new τ_2-expansionary stage later, then U_2 has a change on y_2 say, which undefines $\Delta_2^{U_2}(s(x_2))$, $\Delta_2^{U_2}(s(x_1))$ and $\Delta_2^{U_2}(z)$. This nested procedure is the core part of the construction of bubbles, and the idea can be generalized to a case when β is working below Δ-outcome of several $\mathcal{R}$-strategies. $\square$

In [3], Arslanov, Kalimullin and Lempp actually proved the existence of 3-bubbles, a generalization of double bubbles.

Definition 3. Let $\mathbf{d}, \mathbf{e}, \mathbf{f}$ be 3-c.e. degrees with $\mathbf{0} < \mathbf{d} < \mathbf{e} < \mathbf{f}$. Say that these degrees form a *3-bubble* in $\mathcal{D}_3$ if any 3-c.e. degree $\mathbf{u} < \mathbf{f}$ is comparable with $\mathbf{e}$ and $\mathbf{d}$. Say that these degrees form a *weak 3-bubble* in $\mathcal{D}_3$ if any 3-c.e. degree $\mathbf{u} < \mathbf{f}$ is either comparable with both $\mathbf{e}$ and $\mathbf{d}$ or incomparable with both of them.

Theorem 11 implies that weak 3-bubbles do not exist in $\mathcal{D}_2$.

In [3], Arslanov, Kalimullin and Lempp actually proved that degrees $\mathbf{f}, \mathbf{e}, \mathbf{d}$ in the weak 3-bubbles can be 3-c.e., d.c.e. and c.e., respectively. In the following, we show that such weak 3-bubbles are actually 3-bubbles, so the construction given in [3] produces a 3-bubble.

First, we show that all d.c.e. degrees below $\mathbf{f}$ are comparable with $\mathbf{e}$ and $\mathbf{d}$. Suppose not, and let $\mathbf{g}$ be a d.c.e. degree below $\mathbf{f}$, but not comparable with $\mathbf{e}$ and $\mathbf{d}$. Then $\mathbf{g} \cup \mathbf{d}$ would be d.c.e. and $\mathbf{g} \cup \mathbf{d} > \mathbf{d}$, which would imply that $\mathbf{g} \cup \mathbf{d} > \mathbf{e}$, as $\mathbf{g}$ is not below $\mathbf{e}$. By assumption that $\mathbf{f}, \mathbf{e}, \mathbf{d}$ form a weak 3-bubble in $\mathcal{D}_3$, we know that $\mathbf{g} \cup \mathbf{d}, \mathbf{e}, \mathbf{d}$ also form a weak 3-bubble in $\mathcal{D}_3$, which is also a weak 3-bubble in $\mathcal{D}_2$, a contradiction.

We now assume that $\mathbf{h}$ is a 3-c.e. degree below $\mathbf{f}$ but incomparable with $\mathbf{e}$ and $\mathbf{d}$. Then $\mathbf{h}$ is a properly 3-c.e. degree, and degrees of Lachlan sets of those 3-c.e. sets in $\mathbf{h}$ are d.c.e., and hence are comparable with $\mathbf{e}$ and $\mathbf{d}$.

Let $\mathbf{u}$ be a degree of the Lachlan set of a 3-c.e. set in $\mathbf{h}$. Then $\mathbf{u}$ is not above $\mathbf{d}$, as otherwise, $\mathbf{h}$ would be also above $\mathbf{d}$, which is impossible. As a consequence, $\mathbf{u}$ is below $\mathbf{d}$. Now consider $\mathbf{h} \cup \mathbf{d}$, which is 3-c.e and relative enumerable in and above $\mathbf{d}$. As here, we assume that $\mathbf{d}$ given as a c.e. degree, by a well-known result of Arslanov, LaForte and Slaman in [33] that the class of the d.c.e. degrees coincides with the intersection of the class of the ω-c.e. degrees and the class of the 2-REA degrees, we know that $\mathbf{h} \cup \mathbf{d}$ is d.c.e. Note that $\mathbf{h} \cup \mathbf{d} > \mathbf{d}$, and hence $\mathbf{h} \cup \mathbf{d}$ is comparable with $\mathbf{e}$. As $\mathbf{h}$ itself is incomparable with $\mathbf{e}$, $\mathbf{h} \cup \mathbf{d}$ is above $\mathbf{e}$. Thus, $\mathbf{h} \cup \mathbf{d}, \mathbf{e}, \mathbf{d}$ form a weak 3-bubble in $\mathcal{D}_3$, which is also a weak 3-bubble in $\mathcal{D}_2$. A contradiction again. This completes the proof.

4. Isolation, cupping and diamond embeddings

In this section, we show how to use isolation phenomenon to provide alternative proofs of several known results of cupping properties and diamond embeddings.

In [11], Cooper, Harrington, Lachlan, Lempp and Soare proved the existence of an incomplete, maximal d.c.e. degree $\mathbf{d}$. This result is strong, as it implies that $\mathbf{d}$ cups every c.e. degree not below it to $\mathbf{0}'$. In contrast, Li, Song and Wu proved in [34] the existence of an incomplete ω-r.e. degree cupping every nonzero r.e. degree to $\mathbf{0}'$. These degrees are said to have the *universal cupping property.* In terms of the Ershov hierarchy, Li, Song and Wu's result is optimal.

In [30], Liu and Wu proposed a cupping property for the d.c.e. degrees, where a d.c.e. degree $\mathbf{d}$ has the almost universal cupping property if it cups every c.e. degree not below it to $\mathbf{0}'$. The maximal d.r.e. degree constructed by Cooper et al. does have this property. However, compared to the construction of incomplete maximal d.r.e. degrees, the construction of d.c.e. degrees with almost universal cupping property is much easier. In [30], Liu and Wu constructed a d.c.e. degree $\mathbf{d}$ with the almost cupping property such that $\mathbf{d}$ is also isolated by a c.e. degree $\mathbf{b} < \mathbf{d}$.

Theorem 13. *There is an almost universal cupping d.c.e. degree* $\mathbf{d}$ *and a c.e. degree* $\mathbf{b} < \mathbf{d}$ *such that* $\mathbf{b}$ *is the greatest c.e. degree below* $\mathbf{d}$.

To prove Theorem 13, we will construct a d.c.e. set D, a nonrecursive c.e. set B such that (i) $D \not\leq_T B$, (ii) for any c.e. set W_e, either $W_e \leq_T B$ or $B \oplus D \oplus W_e \equiv_T \emptyset'$. That is, the constructed sets need to satisfy the isolation requirements and also the following cupping requirements:

$\mathcal{R}_e$: $K = \Gamma_e^{B,D,W_e} \vee W_e = \Delta_e^B$, where Γ_e and Δ_e are p.c. functionals constructed by us.

Here K is a fixed creative set. Note that the $\mathcal{R}$-requirements ensure that $B \oplus D$ has the almost universal cupping property.

Let β be an $\mathcal{R}_e$-strategy. For convenience, we write Γ_β for $\Gamma_{e(\beta)}$ and W_β for $W_{e(\beta)}$. β will construct a partial computable (p.c.) functional Γ_β such that $K = \Gamma_\beta^{B,D,W_\beta}$, and if β fails, due to the actions of the isolation strategies, then an isolation strategy will show that $W_\beta \leq_T B$.

Γ_β is constructed as follows:

A. At a stage s, define $\Gamma_\beta^{B,D,W_\beta}(z)[s] = K_s(z)$ for those $z < s$ with $\Gamma_\beta^{B,D,W_\beta}(z)[s]$ not defined, and the use $\gamma_\beta(z)[s]$ is selected as a fresh number.

B. If $\Gamma_\beta^{B,D,W_\beta}(z)[s] \downarrow \neq K_s(z)$, then we put $\gamma_\beta(z)[s]$ into D to undefine the current $\Gamma_\beta^{B,D,W_\beta}(z)$ for the least $z < s$.

In the construction, to correct Γ, β may enumerate uses $\gamma_\beta(z)$ into D for infinitely many z. These enumerations can cause direct conflicts between β and those isolation strategies, η say, below β, which want to preserve computations. This type of interaction is an important component of the whole construction.

Let η be an isolation strategy. The basic idea of η is to construct a p.c. functional Θ_η at expansionary stages to ensure that if $\Phi_\eta^{B,D}$ is total, then Θ_η is well-defined and computes W_η correctly. If later, at an η-expansionary stage, we see that $\Theta_\eta^B(y)$ and $W_\eta(y)$ differ at an argument y say, we will then force a disagreement between $\Phi_\eta^{B,D}(y) \neq W_\eta(y)$.

η has three outcomes f, d and ∞, with priority $\infty <_L f <_L d$. Here f denotes that there are only finitely many η-expansionary stages and η does not create any disagreement, and ∞ denotes that there are infinitely many η-expansionary stages. d denotes the outcome that η succeeds in creating a disagreement between $\Phi_\eta^{B\oplus D}$ and W_η.

A crucial action of η is that when a number, z say, is removed from D, then another number, for example, the stage when z is put into D, is enumerated into B simultaneously. This action can ensure that all the isolation strategies work consistently.

We now consider the interaction between one isolation strategy and one $\mathcal{R}$-strategy. Let β and η be an $\mathcal{R}$-strategy and an isolation strategy respectively, with $\beta \subset \eta$. As mentioned in a single $\mathcal{R}$-strategy, a disagreement created by η could be destroyed by β's enumerations into D. Also, when η has ∞ as its outcome, β may enumerate $\gamma(n)$ into D as n enters K. Now η may see an opportunity to diagonalize by extracting $\gamma(n)$ from D, and η cannot do this as β would be injured by this extraction.

To avoid this, when η sees a computation $\Phi_\eta^{B\oplus D}(y)$ and wants to preserve it, η needs to make this computation clear of the γ_β-uses, by applying the "capricious method", an argument first introduced by Lachlan in his nonsplitting theorem. That is, when η is first visited, it picks a number k_η as its threshold, and whenever a number $k \leq k_\eta$ enters K, we enumerate the current $\gamma_\beta(k)$-use into D to undefine $\Gamma_\beta^{B,D,W_\eta}(k)$, and also reset η by cancelling all the parameters associated to η, except for the parameter k_η.

η aims to define a p.c. functional $\Delta_{\eta\beta}^B$ with the purpose that if η cannot satisfy the associated isolation requirement, then $\Delta_{\eta\beta}^B$ should be total and computes W_β correctly. This will satisfy the $\mathcal{R}_\beta$-requirement.

Suppose that after stage s, η is not reset and suppose that at a stage $t > s$, η sees a potential witness y for its disagreement argument, then η puts $\gamma_\beta(k_\eta)[t]$ into D first, to start an attack on β, by defining

$$\Delta_{\eta\beta}^B \restriction \gamma_\beta(k_\eta)[t] = W_{\beta,t} \restriction \gamma_\beta(k_\eta)[t]$$

with use t.

If W_β changes below $\gamma_\beta(k_\eta)[t]$ after stage t, at an η-expansionary stage $t' > t$ say, then η performs the disagreement argument by removing numbers out of D, including $\gamma_\beta(k_\eta)[t]$, to recover computation $\Phi_\eta^{B\oplus D}(y)$ to $\Phi_\eta^{B\oplus D}(y)[s']$, where s' is the stage at which $\Theta_\eta^B(y)$ is defined, as indicated above. This W_β-change lifts the value of $\gamma_\beta(z)$ for those $z \geq k_\eta$, and hence, after stage t', the enumeration of the γ_β-uses will not affect the computation $\Phi_\eta^{B\oplus D}(x)$. That is, the attack is completed at stage t', and η passes the threshold k_η for β.

On the other hand, if W_β has no changes below $\gamma_\beta(k_\eta)[t]$ after stage t, then the attack associated with $\gamma_\beta(k_\eta)[t]$ keeps active until a new attack is activated. If infinitely many such attacks are started, then $\Delta_{\eta\beta}^B$ is defined as a total function and computes W_β correctly, and hence W_e is computable in B.

In this situation, η has four possible outcomes:

- f : There are only finitely many η-expansionary stages.
- d : η passes its threshold k_η for β, and a disagreement is created.
- ∞ : There are infinitely many η-expansionary stages, and only finitely many attacks are started. In this case, Θ_η^B is total and computes W_η correctly.
- g_β : Infinitely many attacks are started in the construction, and η never passes its threshold k_η for β. $\Delta_{\eta\beta}^B$ is total and computes W_β correctly. The $\mathcal{R}_\beta$-requirement is satisfied. *In this case,* $\Gamma_\beta^{B,D,W_\beta}(p_\eta)$ *diverges.*

Let ξ be any strategy below the outcome g_β, then ξ knows that $\gamma_\beta(p_\eta)$-uses goes to infinity, and we say that a computation $\Phi_\xi^{B\oplus D}(y)$ at a stage s is ξ-believable if $\gamma_\beta(p_\eta)[s]$ is bigger than the use $\varphi_\xi(y)[s]$. If ξ is a back-up strategy for η, then by using only ξ-believable computations, ξ can satisfy the corresponding requirement in the standard way, as after ξ sees at ξ-believable computations, β's further enumerations into D will not affect these computations.

This basic idea can be generalized to the situation when one isolation strategy is working below several $\mathcal{R}$-strategies, where an attack of η needs to pass several thresholds. Please refer to [30] for further development. In [30], Liu and Wu also proved that $\mathbf{b}$ can be cappable. This implies that any d.c.e. degree below $\mathbf{b}$ and any d.c.e. degree above $\mathbf{d}$, together with $\mathbf{0}$ and $\mathbf{0}'$, form a diamond.

This isolation feature allows Fang, Liu and Wu to improve a result of Downey, Li and Wu in [35]. Fang, Liu and Wu proved recently that for any nonzero cappable c.e. degree $\mathbf{c}$, there is a d.c.e. degree $\mathbf{d}$ with almost universal cupping property and a c.e. degree $\mathbf{b} < \mathbf{d}$ such that $\mathbf{b}$ isolates $\mathbf{d}$ and that $\mathbf{c}$ and $\mathbf{b}$ form a minimal pair. By applying this result twice, first to $\mathbf{c}$ and then to $\mathbf{b}$, we have $\mathbf{d}$ and $\mathbf{b}$ first, and then $\mathbf{e}$ and $\mathbf{a}$ such that $\mathbf{e}$ has almost universal cupping property and $\mathbf{a} < \mathbf{e}$ isolates $\mathbf{e}$, and $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair. Now for any nonzero c.e. degree $\mathbf{w}$, $\mathbf{w}$ cups either $\mathbf{e}$ or $\mathbf{d}$, or both, to $\mathbf{0}'$. Obviously, this result has Li–,Yi cupping theorem as a direct corollary.

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Резюме

Г. Ву, М.М. Ямалеев. Изолированность: обоснования и приложения.

В статье рассматриваются феномен изолированной степени и его приложения к исследованию свойств степеней их иерархии Ершова. Анализируются степени, образующие «восьмерку» (более сильный вариант изолированной степени), а также демонстрируются последние достижения в изучении вложимости решеток при помощи свойства изолированности.

Ключевые слова: тьюринговые степени, иерархия Ершова, изолированные степени, вложения решеток.

References

1. *Cooper S.B., Yi X.* Isolated d.r.e. degrees // Preprint series (University of Leeds, Dept. of Pure Math.). – 1995. – No 17. – 25 p.
2. *LaForte G.* Isolation in the *CEA* hierarchy // Arch. Math. Logic. – 2005. – V. 44, No 2. – P. 227–244.
3. *Arslanov M., Kalimullin I., Lempp S.* On Downey's conjecture // J. Symbolic Logic. – 2010. – V. 75, No 2. – P. 401–441.
4. *Cai M., Shore R.A., Slaman T.A.* The n-r.e. degrees: undecidability and Σ_1 substructures // J. Math. Logic. – 2012. – V. 12, No 1. – P 1250005-1–1250005-30.
5. *Wu G.* Structural Properties of d.c.e. degrees and presentations of c.e. reals: PhD Thesis. – Wellington: Victoria University of Wellington, 2002.
6. *Fang C., Liu J., Wu G.* Cupping and diamond embeddings: a unifying approach // Lecture Notes in Computer Science. – 2011. – V. 6735. – P. 71–80.

7. *Soare R.I.* Recursively enumerable sets and degrees. – Heidelberg: Springer-Verlag, 1987. – 437 p.

8. *Cooper S.B.* Local degree theory // Handbook of Computability Theory (Studies in Logic and the Foundations of Mathematics, V. 140) / Ed. by E.R. Griffor. – Amsterdam: North-Holland, 1999. – P. 121–153.

9. *Arslanov M.* Open questions about the n-c.e. degrees // M. Lerman, P.A. Cholak, S. Lempp, R.A. Shore (Eds.) Computability Theory and Its Applications: Current Trends and Open Problems: Proc. 1999 AMS-IMS-SIAM Joint Summer Res. Conf. (Contemporary Mathematics, V. 257). – Providence, RI: Amer. Math. Soc., 2000. – P. 15–22

10. *Cooper S.B.* Degrees of Unsolvability: PhD Thesis. – Leicester: University of Leicester, 1971.

11. *Cooper S.B., Harrington L., Lachlan A.H., Lempp S., Soare R.I.* The d.r.e. degrees are not dense // Ann. Pure Appl. Logic. – 1991. – V. 55., No 2. – P. 125–151.

12. *Arslanov M.* Structural properties of the degrees below $0'$ // Dokl. Acad. Nauk SSSR. – 1985. – V. 283. – P. 270–273.

13. *Downey R.* D.r.e. degrees and the nondiamond theorem // Bull. London Math. Soc. – 1989. – V. 21. – P. 43–50.

14. *Ishmukhametov Sh.* D.r.e. sets, their degrees and index sets: PhD Thesis. – Novosibirsk, 1986.

15. *Kaddah D.* Infima in the d.r.e. degrees // Ann. Pure Appl. Logic. – 1993. – V. 62. – P. 207–263.

16. *Ding D., Qian L.* Isolated d.r.e. degrees are dense in r.e. degree structure // Arch. Math. Logic. – 1996. – V. 36, No 1. – P. 1–10.

17. *LaForte G.* The isolated d.r.e. degrees are dense in the r.e. degrees // Math. Logic Quart. – 1996. – V. 42. – P. 83–103.

18. *Arslanov M.M., Lempp S., Shore R.A.* On isolating r.e. and isolated d-r.e. degrees // Computability, enumerability, unsolvability (London Mathematical Society, Lecture Note Series, No. 224) / Ed. by S.B. Cooper, T.A. Slaman, S.S. Wainer. – Cambridge: Cambridge Univ. Press, 1996. – V. 224. – P. 61–80.

19. *Ishmukhametov Sh., Wu G.* Isolation and the high/low hierarchy // Arch. Math. Logic. – 2002. – V. 41, No 3. – P. 259–266.

20. *Li A., Wu G., Yang Y.* Bounding computably enumerable degrees in the Ershov hierarchy // Ann. Pure Appl. Logic. – 2006. – V. 141, No 1–2. – P. 79–88.

21. *Cooper S.B.* Minimal pairs and high recursively enumerable degrees // J. Symbolic Logic. – 1974. – V. 39, No 4. – P. 655–660.

22. *Chong C.T., Li A., Yang Y.* The existence of high nonbounding degrees in the difference hierarchy // Ann. Pure Appl. Logic. – 2006. – V. 138, No 1–3. – P. 31–51.

23. *Cooper S.B., Salts M.C., Wu G.* The non-isolating degrees are upwards dense in the computably enumerable degrees // Theory and Applications of Models of Computation, Proc. 5th Int. Conf. TAMC 2008 (Lecture Notes in Computer Science, V. 4978). – Berlin; Heidelberg: Springer-Verlag, 2008. – P. 588–596.

24. *Salts M.* An interval of computably enumerable isolating degrees // Math. Logic Quart. – 1999. – V. 45. – P. 59–72.

25. *Lachlan A.H.* Lower bounds for pairs of recursively enumerable degrees // Proc. London Math. Soc. – 1966. – V. 16. – P. 537–569.

26. *Yang Y., Yu L.* On Σ_1-structural differences among finite levels of the Ershov hierarchy // J. Symbolic Logic. – 2006. – V. 71, No 4. – P. 1223–1236.

27. *Wu G.* On the density of the pseudo-isolated degrees // Proc. London Math. Soc. – 2004. – V. 88, No 2. – P. 273–288.

28. *Ishmukhametov Sh.* On the r.e. predecessors of d.r.e. degrees // Arch. Math. Logic. – 1999. – V. 38, No 6. – P. 373–386.

29. *Wu G.* Isolation and lattice embeddings // J. Symbolic Logic. – 2002. – V. 67, No 3. – P. 1055–1064.

30. *Liu J., Wu G.* An almost-universal cupping degree // J. Symbolic Logic. – 2011. – V. 76, No 4. – P. 1137–1152.

31. *Cooper S.B., Li A.* Splitting and cone avoidance in the d.c.e. degrees // Sci. China, Ser. A. – 2002. – V. 45, No 9. – P. 1135–1146.

32. *Yamaleev M.M.* Splitting in 2-computably enumerable degrees with avoiding cones // Russian Mathematics (Izv. Vuz. Mat.). – 2009. – V. 53, No 6. – P. 63–66.

33. *Arslanov M., LaForte G., Slaman T.* Relative enumerability in the difference hierarchy // J. Symbolic Logic. – 1998. – V. 63, No 2. – P. 411–420.

34. *Li A., Song Y., Wu G.* Universal cupping degrees // Theory and Applications of Models of Computation, Proc. 3rd Int. Conf. TAMC 2006 (Lecture Notes in Computer Science, V. 3959). – Berlin; Heidelberg: Springer-Verlag, 2006. – P. 721–730.

35. *Downey R.G., Li A., Wu G.* Complementing cappable degrees in the difference hierarchy // Ann. Pure Appl. Logic. – 2004. – V. 125, No 1–3. – P. 101–118.

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