Distance-regular graphs of diameter 4

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Amply regular graph

We consider undirected graphs without loops amd multiple edges. For vertex u of a graph Γ the subgraph $\Gamma_i(u) = \{w \mid d(u,w) = i\}$ is called i-neighboorhood of u in Γ. We set $[u] = \Gamma_1(u)$, $u^\perp = \{u\} \cup [u]$.

Degree of an vertex a of Γ is the number of vertices in $[a]$.

Graph Γ is called regular of degree k, if the degree of any vertex is equal k. The graph Γ is called amply regular with parameters (v,k,λ,μ) if Γ is regular of degree k on v vertices, and $|[u] \cap [w]|$ is equal λ, if u adjacent to w, is equal μ, if $d(u,w) = 2$. Amply regular graph of diameter 2 is called strongly regular.

Distance-regular graph

If $d(u,w)=i$ then by $b_i(u,w)$ (by $c_i(u,w)$) we denote the number of vertices in $\Gamma_{i+1}(u)\cap[w]$ (in $\Gamma_{i-1}(u)\cap[w]$). The graph Γ with diameter d is called distance-regular with intersection array $\{b_0,b_1,...,b_{d-1};c_1,...,c_d\}$ if $b_i=b_i(u,w)$ and $c_i=c_i(u,w)$ for every $i\in\{0,...,d\}$ and for every two vertices u,w with $d(u,w)=i$ (see [1]). Distance-regular graph of diameter 2 is called strongly regular with parameters (v,k,λ,μ), where v is the number of vertices of the graph, $k=b_0$, $\lambda=k-b_1-1$ and $\mu=c_2$.

Intersection numbers

Let Γ be a distance-regular graph of diameter d with v vertices. Then we have the symmetric association scheme $(X, \mathcal{R})$ with d classes, where X is the set of vertices of Γ and $R_i = \{(u,w) \in X^2 \mid d(u,w) = i\}$. For vertex $u \in X$ set $k_i = |\Gamma_i(u)|$. Let A_i be the adjacency matrix of the graph Γ_i. Then $A_i A_j = \sum p^l_{ij} A_l$ for some integer numbers $p^l_{ij} \geq 0$, which are called the intersection numbers. Note that $p^l_{ij} = |\Gamma_i(u) \cap \Gamma_j(w)|$ for every two vertices u, w with $d(u,w) = l$.

Some new graphs

Let Γ be a distance-regular graph of diameter d and $i, j \in \{1, 2, ..., d\}$. The graph Γ_i has $V(\Gamma_i) = V(\Gamma)$ and vertices u, w are adjacent in Γ_i if and only if $d_\Gamma(u, w) = i$. The graph $\Gamma_{i,j}$ has $V(\Gamma_{i,j}) = V(\Gamma)$ and vertices u, w are adjacent in Γ_i if and only if $d_\Gamma(u, w) \in \{i, j\}$.

Strongly regular graphs

The common properties of strongly regular graphs are in

Proposition 1

Let Γ be a strongly regular graph with parameters (v, k, λ, μ). Then $(v-k-1)\mu = k(k-\lambda-1)$ and one of the following holds:

1. $k = 2\mu$, $\lambda = \mu - 1$ and $v = 4\mu + 1$ is the sum of two squares of some integers;
2. $(\lambda - \mu)^2 + 4(k - \mu)$ is the square of some positive integer n, and Γ has spectrum k^1, r^f, s^{v-f-1}, where $r = (\lambda - \mu + n)/2$, $s = (\lambda - \mu - n)/2$ and $f = (s+1)k(s-k)/(n\mu)$.

Partial geometries

Partial geometry $pG_\alpha(s,t)$ is a geomerty of points and lines such that every line has $s+1$ points, every point is on $t+1$ lines (with $s>0,\ t>0$) and for any antiflag (P,y) there is α lines z_i containing P and intersecting y. In the case $\alpha=1$ we have generalized quadrangle $GQ(s,t)$.

Pseudo-geometric graph

Point graph of the partial geometry $pG_\alpha(s,t)$ has points as vertices and two points are adjacent if its belong to some line. Point graph of the partial geometry $pG_\alpha(s,t)$ is strongly regular with parameters $v=(s+1)(1+st/\alpha)$, $k=s(t+1)$, $\lambda=s-1+(\alpha-1)t$, $\mu=\alpha(t+1)$. Strongly regular graph with this parameters for some natural numbers s,t,α is called pseudo-geometric graph for $pG_\alpha(s,t)$. This graph has nonprincipal eigenvalues $s-\alpha$ and $-(t+1)$.

Eigenvalues of regular graphs

Distance-regular graph of diameter d has exactly $d+1$ eigenvalues $\theta_0 = k > \theta_1 > ... > \theta_d$.

Theorem 1 [2]

Let Γ be a distance-regular graph with valency k at least three and diameter d at least three. Then the following hold:

1. $\theta_d < (a_1 - (a_1^2 + 4k)^{1/2})/2$;
2. $\theta_1 \geq \min\{(a_1 + (a_1^2 + 4k)^{1/2})/2, a_3\}$;
3. if $d \geq 4$, then $\theta_1 \geq (a_1 + (a_1^2 + 4k)^{1/2})/2$.

Bounds for θ_1, θ_d

fundamental bound

In Jurishich et al. [3], it was shown that for a distance-regular graph with diameter d at least two one has the following fundamental bound:

$$(\theta_1 + \frac{k}{a_1+1})(\theta_d + \frac{k}{a_1+1}) \geq -\frac{ka_1b_1}{(a_1+1)^2}.$$

Let

$$b^+ = -1 - \frac{b_1}{1+\theta_d},\ b^- = -1 - \frac{b_1}{1+\theta_1}.$$

Nonbipartite graph with equality in fundamental bound is called tight graph. Local subgraph in tight graph is strongly regular with eigenvalues a_1, b^+, b^-. Fundamental bound have also the following shape $k(a_1 + b^+b^-) \leq (a_1 - b^+)(a_1 - b^-)$.

AT4(p,q,r)-graphs

Tight antipodal graph of diameter 3 is a Taylor graph with intersection array $\{k, \mu, 1; 1, \mu, k\}$.
Antipodal graph Γ of diameter 4 has intersection array $\{k, k - a_1 - 1, (r - 1)c_2, 1; 1, c_2, k - a_1 - 1, k\}$ (Proposition 4.2.2 [1]). The graph Γ is tight if and only if $q^4_{11} = 0$ [3]. In this case every local subgraph is strongly regular with eigenvalues $a_1, p = b^+, -q = b^-$ and all parameters of Γ expressed by p, q, r, where r is the antipodality index (the size of antipodality class). So Γ is called AT4(p,q,r)-graph.

Inverse problem

Let Γ be a distance-regular graph of diameter 4. If $\Gamma_{3,4}$ is strongly regular graph then a founding the intersection array of Γ by parameters of $\Gamma_{3,4}$ is the inverse problem.

The known examples of primitive graphs.

1. Odd graph $\Gamma = O_9$ has intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$ and $\Gamma_{3,4}$ has parameters (126,100,78,84).
2. Folded 9-cube Γ has intersection array $\{9, 8, 7, 6; 1, 2, 3, 4\}$ and $\Gamma_{3,4}$ has parameters (256,210,170,182).
3. Dual polar graph Γ has intersection array $\{30, 28, 24, 16; 1, 3, 7, 15\}$ and $\Gamma_{3,4}$ has parameters (2295,1984,1708,1860).

Inverse problem for antipodal graphs

There are the unique bipartite antipodal graph Γ with strongly regular graph $\Gamma_{3,4}$ (p. 425 [1]):

4. 4-cube Γ has intersection array $\{4,3,2,1;1,2,3,4\}$ and $\Gamma_{3,4}$ has parameters (16,5,0,2).

Note that AT4(p,q,r)-graph Γ has strongly regular graph $\Gamma_{3,4}$ if and only if $r=2$ and $q=p+2$.

Theorem 1 [4]. AT4(p,q,r)-graph with $r=2$ and $q=p+2$ does not exist.

Theorem 2 [5]. Let Γ be an antipodal distance-regular graph of diameter 4 with strongly regular graph $\Delta=\Gamma_{3,4}$. Then $\lambda(\Delta)=0$, $b_0=k(\Delta)-1$, $c_2=a_1+2=\mu(\Delta)$ and $b_1=k(\Delta)-\mu(\Delta)$.

Corollary 1 [5]

Antipodal graph Γ with intersection array
$\{56, 45, 12, 1; 1, 12, 45, 56\}$ ($\Gamma_{3,4}$ has parameters (324,57,0,12)),
$\{115, 96, 20, 1; 1, 20, 96, 115\}$ ($\Gamma_{3,4}$ has parameters (784,116,0,20)),
$\{204, 175, 30, 1; 1, 30, 175, 204\}$ ($\Gamma_{3,4}$ has parameters (1600,205,0,30)) or
$\{329, 288, 42, 1; 1, 42, 288, 329\}$ ($\Gamma_{3,4}$ has parameters (2916,330,0,42))
do not exist.

Small antipodal graphs

Small antipodal graph Γ with strongly regular graph $\Gamma_{3,4}$ has intersection array [6]:

5. $\{20, 18, 3, 1; 1, 3, 18, 20\}$, $\Gamma_{3,4}$ has parameters (162,21,0,3).
6. $\{25, 24, 2, 1; 1, 2, 24, 25\}$, $\Gamma_{3,4}$ has parameters (352,26,0,2).
7. $\{32, 27, 6, 1; 1, 6, 27, 32\}$, $\Gamma_{3,4}$ has parameters (210,33,0,6).
8. $\{36, 35, 2, 1; 1, 2, 35, 36\}$, $\Gamma_{3,4}$ has parameters (704,37,0,2).
9. $\{45, 40, 6, 1; 1, 6, 40, 45\}$, $\Gamma_{3,4}$ has parameters (392,46,0,6).
10. $\{49, 48, 2, 1; 1, 2, 48, 49\}$, $\Gamma_{3,4}$ has parameters (1276,50,0,2).
11. $\{54, 50, 5, 1; 1, 5, 50, 54\}$, $\Gamma_{3,4}$ has parameters (650,55,0,5).
12. $\{56, 45, 12, 1; 1, 12, 45, 56\}$, $\Gamma_{3,4}$ has parameters (324,57,0,12).

Small antipodal graphs

13. $\{75, 64, 12, 1; 1, 12, 64, 75\}$, $\Gamma_{3,4}$ has parameters (552,76,0,12).
14. $\{77, 72, 6, 1; 1, 6, 72, 77\}$, $\Gamma_{3,4}$ has parameters (1080,78,0,6).
15. $\{81, 80, 2, 1; 1, 2, 80, 81\}$, $\Gamma_{3,4}$ has parameters (3404,82,0,2).
16. $\{84, 75, 10, 1; 1, 10, 75, 84\}$, $\Gamma_{3,4}$ has parameters (800,85,0,10).
17. $\{96, 91, 6, 1; 1, 6, 91, 96\}$, $\Gamma_{3,4}$ has parameters (1650,97,0,6).
18. $\{117, 112, 6, 1; 1, 6, 112, 117\}$, $\Gamma_{3,4}$ has parameters (2420,118,0,6).
19. $\{135, 128, 8, 1; 1, 8, 128, 135\}$, $\Gamma_{3,4}$ has parameters (2432,136,0,8).
20. $\{140, 126, 15, 1; 1, 15, 126, 140\}$, $\Gamma_{3,4}$ has parameters (1458,141,0,15).

Theorems on small antipodal graphs

Theorem 3 [5]. If distance-regular graph with intersection array $\{32, 27, 12(r-1)/r, 1; 1, 12/r, 27, 32\}$ exist then $r = 3$.
The uniquality of graph with intersection array $\{32, 27, 8, 1; 1, 4, 27, 32\}$ it is proved by Soicher in [7].
Theorem 4 [5]. Let distance-regular graph with intersection array $\{56, 45, 24(r-1)/r, 1; 1, 24/r, 27, 32\}$, $r \in \{2, 3, 4, 6, 8\}$ exist. Then $r = 3$.
Theorem 5 [5]. If distance-regular graph Γ with intersection array $\{96, 75, 32(r-1)/r, 1; 1, 32/r, 75, 96\}$ exist, then $r = 2$, Γ is not locally $GQ(5, 3)$-graph and the group $G = \mathrm{Aut}(\Gamma)$ acts intrasitively on the set of antipodal classes of Γ.

Triple intersection numbers

Let Γ be a distance-regular graph of diameter d.
If u_1, u_2, u_3 be a vertices of Γ, and r_1, r_2, r_3 be integers from $\{0, 1, ..., d\}$ then we define $\left\{ \begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right\}$ as a set vertices $w \in \Gamma$ such that $d(w, u_i) = r_i$, and $\left[\begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right]$ is the number of vertices in $\left\{ \begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right\}$.
The numbers $\left[\begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right]$ are called triple intersection numbers.
We abbreviate the latter as $[r_1 r_2 r_3]$ whenever no confusion about the triple (u_1, u_2, u_3) may arise.

Unlike for the case $t = 2$, for $t \geq 3$ there are no formulas for $[r_1 r_2 r_3]$ that are generally valid in the case of distance-regular graphs. However, certain restrictions for their values may be found [8]. Let u, v, w be three fixed vertices in Γ, and let $W = d(u, v), U = d(v, w), V = d(u, w)$.

Triple intersection numbers

There exists precisely one vertex $x = u$ such that $d(x, u) = 0$, so $[0jh]$ is either 0 or 1. We can apply the same argument also for v and w. Altogether, we obtain

$$[0jh] = \delta_{jW}\delta_{hV}, [i0h] = \delta_{iW}\delta_{hU}, [ij0] = \delta_{iU}\delta_{jV} (0 \leq i, j, h \leq 3).$$

Another set of equations can be obtained by fixing the distance from two of the vertices u, v, w and counting vertices at all distances from the third vertex:

Triple intersection numbers

$$\sum_{l=1}^{d}[ljh] = p^U_{jh} - [0jh],$$

$$\sum_{l=1}^{d}[ilh] = p^V_{ih} - [i0h], \qquad (+)$$

$$\sum_{l=1}^{d}[ijl] = p^W_{ij} - [ij0].$$

Triple intersection numbers

We can use the triangle inequality to conclude vanishing of some variables. For example, for $0 \le i, j \le 3$ and $|i-j| > W$ or $i+j < W$ we have $p_{ij}^W = 0$ and so also $[ijh] = 0 (0 \le h \le 3)$.
If a Krein parameter q_{ij}^h is zero, we can obtain another equation for triple intersection numbers.
Define $S_{ijh}(u,v,w) = \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th} \left[\begin{smallmatrix} uvw \\ rst \end{smallmatrix} \right]$. If $q_{ij}^h = 0$ then $S_{ijh}(u,v,w) = 0$.

Symmetrization

We fix some vertices u, v, w of distance-regular graph Γ with diameter 3 and set $\{ijh\} = \left\{ \begin{matrix} uvw \\ ijh \end{matrix} \right\}$, $[ijh] = \left[\begin{matrix} uvw \\ ijh \end{matrix} \right]$,
$[ijh]' = \left[\begin{matrix} uwv \\ ihj \end{matrix} \right]$, $[ijh]^* = \left[\begin{matrix} vuw \\ jih \end{matrix} \right]$ $[ijh]^\sim = \left[\begin{matrix} wvu \\ hji \end{matrix} \right]$.
In the cases $d(u,v) = d(u,w) = d(v,w) = 2$ or $d(u,v) = d(u,w) = d(v,w) = 3$ calculation parameters $[ijh]' = \left[\begin{matrix} uwv \\ ihj \end{matrix} \right]$, $[ijh]^* = \left[\begin{matrix} vuw \\ jih \end{matrix} \right]$ and $[ijh]^\sim = \left[\begin{matrix} wvu \\ hji \end{matrix} \right]$ (symmetrization triple intersection numbers array) can to give new equalities and to prove the nonexistence of the graph.

Graphs with $a_4 = 0$

Intersection arrays of distance-regular graphs with $\lambda = 2$ and at most 4096 vertices are founded in [9]. There is array $\{21, 18, 12, 4; 1, 1, 6, 21\}$. Automorphisms of a graph with this array are determined in [10].
A. Brouwer [1, p. 148] suggested

Problem.

Does there exist primitive distance-regular graph of diameter 4 with $a_4 = 0$, apart from Livingstone graph with array $\{11, 10, 6, 1; 1, 1, 5, 11\}$)?

There is noted that arrays $\{21, 18, 12, 4; 1, 1, 6, 21\}$ and $\{22, 21, 21, 3; 1, 1, 3, 22\}$ are feasible and have $a_4 = 0$.

$\{m(2m+1),(m-1)(2m+1),m^2,m;1,m,m(m-1),m(2m+1)\}$

It is known infinite series feasible intersection arrays with $a_4=0$:

$$\{m(2m+1),(m-1)(2m+1),m^2,m;1,m,m(m-1),m(2m+1)\}.$$

It is formally selfdual graphs with $v=8\mu^2(\mu+1)$. In [11] using multiplicity of eigenvalues it is proved tat graph with intersection array

$$\{m(2m+1),(m-1)(2m+1),m^2,m;1,m,m(m-1),m(2m+1)\}$$

does not exist.
We have a new proof of this fact in [12] by using triple intersection numbers.

$\{m(2m+1),(m-1)(2m+1),m^2,m;1,m,m(m-1),m(2m+1)\}$

Let Γ be a distance-regular graph with intersection array

$\{m(2m+1),(m-1)(2m+1),m^2,m;1,m,m(m-1),m(2m+1)\}$.

Let u,v,w be a vertices of Γ, $\{rst\}=\left\{ {uvw \atop rst} \right\}$ and $[rst]=\left[{uvw \atop rst} \right]$. If $d(u,v)=d(u,w)=d(v,w)=1$, then formulas $(+)$ give $[111]=2m/(m+1)$ and graph with array $\{2m^2+m,2m^2-m-1,m^2,m;1,m,m^2-m,2m^2+m\}$ does not exist.

Brouwer problem

Theorem 6 [12].

Distance-regular graph with intersection array $\{21, 18, 12, 4; 1, 1, 6, 21\}$ does not exist.

Theorem 7 [12].

Distance-regular graph with intersection array $\{22, 21, 21, 3; 1, 1, 3, 22\}$ does not exist.

Antipodal Q-polynomial graphs

Antipodal Q-polynomial graph Γ of degree at most 1000 with strongly regular graph $\Gamma_{3,4}$ has intersection array [6]:

1. $\{45, 32, 9, 1; 1, 9, 32, 45\}$, spectrum $45^1, 15^{21}, 3^{90}, -3^{105}, -9^{35}$ and $v = 1 + 45 + 160 + 45 + 1 = 252$.
2. $\{56, 45, 12, 1; 1, 12, 45, 56\}$, spectrum $56^1, 14^{36}, 2^{140}, -4^{126}, -16^{21}$ and $v = 1 + 56 + 210 + 56 + 1 = 324$.
3. $\{96, 75, 16, 1; 1, 16, 75, 96\}$, spectrum $96^1, 24^{46}, 4^{252}, -4^{276}, -16^{69}$ and $v = 1 + 96 + 450 + 96 + 1 = 644$.
4. $\{115, 96, 20, 1; 1, 20, 96, 115\}$, spectrum $115^1, 23^{70}, 3^{345}, -5^{322}, -25^{46}$ and $v = 1 + 115 + 552 + 115 + 1 = 784$.

Q-polynomial graphs

5. $\{117, 80, 18, 1; 1, 18, 80, 117\}$, spectrum
$117^1, 39^{27}, 9^{182}, -3^{351}, -9^{195}$ and
$v = 1 + 117 + 780 + 117 + 1 = 1134$, $AT4(9, 3, 2)$-graph.
6. $\{175, 144, 25, 1; 1, 25, 144, 175\}$, spectrum
$175^1, 35^{247}, 23^{455}, -5^{1729}, -25^{119}$ and
$v = 1 + 175 + 2520 + 175 + 1 = 3400$.
7. $\{176, 135, 24, 1; 1, 24, 135, 176\}$, spectrum
$176^1, 44^{56}, 8^{440}, -4^{616}, -16^{231}$ and
$v = 1 + 176 + 990 + 176 + 1 = 1344$.
8. $\{189, 128, 27, 1; 1, 27, 128, 189\}$, spectrum
$189^1, 63^{29}, 15^{231}, -3^{609}, -9^{406}$ and
$v = 1 + 189 + 896 + 189 + 1 = 1276$.
9. $\{204, 175, 30, 1; 1, 30, 175, 204\}$, spectrum
$204^1, 34^{120}, 4^{714}, -6^{680}, -36^{85}$ and
$v = 1 + 204 + 1190 + 204 + 1 = 1600$.

Q-polynomial graphs

10. $\{261, 176, 54, 1; 1, 54, 176, 261\}$, spectrum $261^1, 87^{30}, 21^{261}, -3^{870}, -9^{638}$ and $v = 1 + 261 + 1276 + 261 + 1 = 1800$, $AT4(21, 3, 2)$-graph.
11. $\{288, 245, 36, 1; 1, 36, 245, 288\}$, spectrum $288^1, 48^{141}, 6^{1080}, -6^{1128}, -36^{188}$ and $v = 1 + 288 + 1960 + 288 + 1 = 2538$, $AT4(6, 6, 2)$-graph.
12. $\{329, 288, 42, 1; 1, 42, 288, 329\}$, spectrum $329^1, 47^{189}, 5^{1316}, -7^{1269}, -49^{141}$ and $v = 1 + 329 + 2256 + 329 + 1 = 3916$.
13. $\{336, 255, 40, 1; 1, 40, 255, 336\}$, spectrum $336^1, 84^{64}, 16^{693}, -4^{1344}, -16^{714}$ and $v = 1 + 336 + 2142 + 336 + 1 = 2816$, $AT4(16, 4, 2)$-graph.
14. $\{414, 350, 45, 1; 1, 45, 350, 414\}$, spectrum $414^1, 69^{162}, 9^{1610}, -6^{1863}, -36^{414}$ and $v = 1 + 414 + 3220 + 414 + 1 = 4050$.

Q-polynomial graphs

15. $\{416, 315, 48, 1; 1, 48, 315, 416\}$, spectrum $416^1, 104^{66}, 20^{780}, -4^{1716}, -16^{1001}$ and $v = 1 + 416 + 2730 + 416 + 1 = 3564$, $AT4(20, 4, 2)$-graph.
16. $\{475, 384, 50, 1; 1, 50, 384, 475\}$, spectrum $336^1, 84^{64}, 16^{693}, -4^{1344}, -16^{714}$ and $v = 1 + 336 + 2142 + 336 + 1 = 2816$, $AT4(15, 5, 2)$-graph.
17. $\{540, 455, 54, 1; 1, 54, 455, 540\}$, spectrum $414^1, 69^{162}, 9^{1610}, -6^{1863}, -36^{414}$ and $v = 1 + 414 + 3220 + 414 + 1 = 4050$, $AT4(12, 6, 2)$-graph.
18. $\{640, 567, 64, 1; 1, 64, 567, 640\}$, spectrum $416^1, 104^{66}, 20^{780}, -4^{1716}, -16^{1001}$ and $v = 1 + 416 + 2730 + 416 + 1 = 3564$, $AT4(8, 8, 2)$-graph.

Existence of Q-polynomial graphs

Graph with intersection array $\{176, 135, 24, 1; 1, 24, 135, 176\}$ exist (it is the first Meixner graph).

Graphs with intersection arrays $\{56, 45, 12, 1; 1, 12, 45, 56\}$, $\{115, 96, 20, 1; 1, 20, 96, 115\}$, $\{204, 175, 30, 1; 1, 30, 175, 204\}$ and $\{329, 288, 42, 1; 1, 42, 288, 329\}$ do not exist [4].
Graphs with intersection arrays $\{45, 32, 9, 1; 1, 9, 32, 45\}$, $\{175, 144, 25, 1; 1, 25, 144, 175\}$, $\{189, 128, 27, 1; 1, 27, 128, 189\}$, $\{414, 350, 45, 1; 1, 45, 350, 414\}$ do not exist [13].

Small antipodal graphs

Let Γ be an antipodal distance-regular graph of diameter 4 wth strongly regular graph $\Delta = \Gamma_{3,4}$ and degree < 32. Then Γ has intersection array $\{20, 18, 3, 1; 1, 3, 18, 20\}$ ($\Gamma_{3,4}$ has parameters (162,21,0,3)) or $\{25, 24, 2, 1; 1, 2, 24, 25\}$ ($\Gamma_{3,4}$ has parameters (352,26,0,2)).

Theorem 8 [13].

Distance-regular graph with intersection array $\{20, 18, 3, 1; 1, 3, 18, 20\}$ does not exist.

Theorem 9 [13].

Distance-regular graph with intersection array $\{25, 24, 2, 1; 1, 2, 24, 25\}$ does not exist.

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