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PROBLEM OF CONJUGATION OF HARMONIC FUNCTIONS IN THREE-DIMENSIONAL DOMAINS WITH NONSMOOTH BOUNDARIES

I.T. Denisyuk

The problems of conjugation of analytic functions in two-dimensional domains with piecewise smooth boundaries were studied in [1], [2]. In this article we consider the problem of construction of harmonic functions in domains of the space R^3 under the condition of their conjugation on surfaces containing non-smooth parts like singular lines (sets of angular point) and conic points.

1. *Statement of problem.* Assume that in R^3 simply connected finite domains are contained V_i, $i = \overline{1, k_0}$, $V_i \cap V_l = \emptyset$, which are bounded by closed surfaces S_i containing smooth closed non-intersecting singular lines (sets of angular points) L_{ij}, $j = \overline{1, p_i}$, and conic points $O_{ik} \notin L_{ij}$, $k = \overline{1, q_i}$.

Let us construct functions $F_i(x, y, z)$ which are harmonic in the corresponding domains V_i, and a function $F_0(x, y, z)$ harmonic in $V_0 = R^3 \setminus \bigcup_{i=1}^{k_0} V_i$ under the following condition of conjugation on the surfaces S_i:

at the points of smoothness $N_0 \in S_i$, $N_0 \notin L_{ij}$, $N_0 \neq O_{ik}$,

$$F_i^+(N_0) - F_0^-(N_0) = 0, \tag{1}$$

$$\lambda_i \partial F_i^+(N_0)/\partial n - \lambda_0 \partial F_0^-(N_0)/\partial n = 0; \tag{2}$$

at the points of singular lines L_{ij} or at conic points O_{ik}

$$\lim[F_i^+(N_0) - F_0^-(N_0)] = 0, \quad \lim[\lambda_i \partial F_i^+(N_0)/\partial n - \lambda_0 \partial F_0^-(N_0)/\partial n] = 0, \tag{3}$$

where the limits are taken as $N_0 \to M_0 \in L_{ij}$ in the normal plane to a singular line, or as $N_0 \to O_{ik}$, the constants λ_0, λ_i belong to R, $\mathbf{n}$ is the exterior normal to S_i. The upper index $\pm$ means the boundary value of the harmonic function in its approach to the surface S_i from the side of the domain V_i (the sign +) or V_0 (the sign −). We assume that near its vertex the conic surface has a rectilinear generatrix and curvilinear smooth closed directrix.

Some problems of the mechanics of continuous media can be reduced to the above problem (see [3], [4]). The harmonicity of the functions $F_i(M)$ at the points of the singular lines L_{ij} or at conic points O_{ik} is understood as the fulfillment of the equality

$$\lim \iiint_{V^*} \nabla^2 F(M) dv = 0, \quad M_0^* \in V^*, \tag{4}$$

where the limits are taken as $V^* \to M_0^*$, i.e., when the domain V^* is contracted to the point M_0^*; from the physical point of view this is conditioned by the necessity to fulfill the laws of energy conservation at singular points of the medium (see [3]–[5]); in addition, $M = M(x, y, z)$, $M_0^* \in L_{ij}$ or $M_0^* = O_{ik}$.

2. *The asymptotic of harmonic functions.* To determine the character of the harmonic functions we study their behavior in local domains of the singular points.

Lemma 1. *If at the points of a singular line L the value*

$$m = \pi/(2\pi - \varpi), \tag{5}$$

where ϖ is the angle of surfaces' opening at the points of the singular line in the plane normal to this line, belongs to the interval $(0,1)$, then the harmonic function and its normal derivative possess the asymptotic representation

$$\begin{gathered} F_i(x,y,z) = \rho^m (C_{i1}\cos m\theta + C_{i2}\sin m\theta) + O(\rho^{m+1}), \\ \partial F_i(x,y,z)/\partial n = \rho^{m-1}(C_{i2} m\cos m\theta - C_{i1} m \sin m\theta) + O(\rho^m), \quad i = 0,1, \end{gathered} \tag{6}$$

where ρ, θ, s are local coordinates, $C_{iq} = C_{iq}(s)$, $q = 1,2$, are values depending on s.

Proof. Assume that the singular line L is the intersection of smooth surfaces S_1 and S_2, $L = S_1 \cap S_2$. Let us expand over L a smooth surface S_0, on which we introduce the orthogonal coordinates u, v so that with $v = \text{const}$ the singular line L is described, and in the result of parameterization we assume $u = s$, where s is the length of the arc of L, counted from a certain initial point (see [6], p. 58). Let $\mathbf{n}\mathbf{n}_1\mathbf{n}_2$ be the moving trihedron of the surface S_0 at the point M_0 of the curve, where $\mathbf{n}_1$, $\mathbf{n}_2$ are unit vectors lying in the plane tangent to S_0 at the point M_0, $\mathbf{n}_1$ is the tangent vector to the curve L, $\mathbf{n} \perp \mathbf{n}_1$, $\mathbf{n} \perp \mathbf{n}_2$. The points M of the plane of vectors $\mathbf{n}$, $\mathbf{n}_2$ are defined by the polar coordinates (ρ, θ), θ stands for the angle formed by the vectors $\mathbf{M}_0\mathbf{M}$ and $\mathbf{n}_2$.

The representation

$$\mathbf{r} = \mathbf{r}_0 + \rho(\cos\theta\,\mathbf{n}_2(s) + \sin\theta\,\mathbf{n}(s)), \tag{7}$$

where $\mathbf{r}_0$ and $\mathbf{r}$ are the radius vectors of the points M_0 and M, respectively, contains the curvilinear orthogonal coordinates ρ, θ, s of the point M of the local domain of the spatial curve L with the Lame coefficients $h_1 = 1$, $h_2 = \rho$, $h_3 = 1 + \rho H_0$, $H_0 = -(p\cos\theta + r_1 \sin\theta)$, $p = -|\mathbf{r}_s|_v/|\mathbf{r}_v|$, $r_1 = (\mathbf{n}, \mathbf{r}_{ss})/|\mathbf{r}_s|$.

The Laplace equation $\Delta F(x,y,z) = 0$ admits the similarity transformation

$$F = A_1 F(B_1 x, B_1 y, B_1 z), \tag{8}$$

where $A_1 = A_1(B_1)$, $A_1, B_1 \in R$. By differentiating relation (8) with respect to B_1, we obtain the equation

$$F(x,y,z)dA_1/dB_1 + A_1(\operatorname{grad} F(x,y,z), \mathbf{r}) = 0,$$

whose solution

$$F(x,y,z) = F_0^*(x,y,z) \tag{9}$$

contains the homogeneous function $F_0^*(x,y,z)$ of m-th order of the variables x, y, z. Passing in (9) to the curvilinear coordinates (7), we establish that the harmonic function belongs to the exponential class with respect to the variable ρ

$$F_i = \rho^m A_i(\theta, s), \quad i = 0,1. \tag{10}$$

By substituting representation (10) into equation (4) written in the coordinates ρ, θ, s, we find the determining equation for the function $A_i(\theta, s)$

$$\partial^2 A_i/\partial\theta^2 + m^2 A_i = 0, \quad m > 0,$$

whose solution gives us the asymptotic representation (6). Satisfying the boundary conditions (3) with the help of (10), we obtain the homogeneous system of linear algebraic equations with respect to the values C_{iq}. The system is solvable if its determinant is equal to zero, which gives us relation (5). □

Assume that the point O is the vertex of a conic surface with the rectilinear generatrices and smooth closed directrix L_0, drawn by the unit vector $\mathbf{r}_0(s_1)$ of the generatrix with the beginning at the point O. Here s_1 is the length of the generatrix counted from a certain initial point. We introduce the local curvilinear coordinates ρ_1, θ_1, s_1 by the equality

$$\mathbf{r}_1 = \rho_1(\mathbf{r}_0(s_1)\sin\theta_1 + \mathbf{n}_3(s_1)\cos\theta_1), \tag{11}$$

where $\mathbf{r}_1$ is the radius vector of the point $M(x,y,z)$, $\mathbf{n}_3(s_1) = [\mathbf{r}_0(s_1), d\mathbf{r}_0(s_1)/ds_1]$ is the normal to the generatrix $\mathbf{r}_0(s_1)$ of the conic surface at the point O, $\rho = |\mathbf{r}_1|$, θ_1 is the angle formed by the vector $\mathbf{r}_1$ and the normal $\mathbf{n}_3(s_1)$. These curvilinear coordinates are orthogonal with the Lame coefficients $h_1 = 1$, $h_2 = \rho_1$, $h_3 = \rho_1 H_{01}$, $H_{01} = \sin\theta_1 + d_1\cos\theta_1$, $d_1 = |[\mathbf{r}_0(s_1), d^2\mathbf{r}_0(s_1)/ds_1^2]|$, and with $\theta_1 = \pi/2$ equation (11) describes the conic surface. As in the proof of Lemma 1, we establish that the harmonic function in a neighborhood of a conic point belongs to the exponential class

$$F(x,y,z) = \rho_1^{m_0} D(\theta_1, s_1). \tag{12}$$

Satisfying equation (4) written in the coordinates ρ_1, θ_1, s_1, by means of (12) we find the equation

$$Q_{m_0}[D(\theta_1, s_1)] = 0, \tag{13}$$

where the operator has the form

$$Q_{m_0}[\cdot] = H_{01}^{-1}(\partial(H_{01}\partial/\partial\theta_1)/\partial\theta_1 + \partial(H_{01}^{-1}\partial/\partial s_1)/\partial s_1) + m_0(m_0+1).$$

The values H_{01}, H_{01}^{-1} are differentiable, periodic with respect to s_{01} functions (s_{01} is the length of the directrix L_0) and in this class we construct the function (see [7], p. 507)

$$D(\theta_1, s_1) = \sum_{n=0}^{K}(D_{n1}(\theta_1)\cos nls_1 + D_{n2}(\theta_1)\sin nls_1) + O(1/K), \tag{14}$$

where $l = 2\pi/s_{01}$.

Substituting segments of the Fourier series for H_0, H_0^{-1}, $D(\theta_1, s_1)$ of the type (14) into equation (13), grouping and equaling to zero the expressions at the harmonics $n = \overline{0,K}$, we obtain a system consisting of $2K+1$ ordinary differential equations of the second order with respect to the unknown functions $D_{nq}(\theta_1)$, $q = 1,2$. This system is compatible (see [8], pp. 12, 135) and has the solution

$$D_{nq}(\theta_1) = \sum_{k=1}^{2(2K+1)} C_{nqk}\Phi_k(\theta_1), \tag{15}$$

where $\Phi_k(\theta_1)$ are bounded and defined for $\theta_1 \in [-\pi,\pi)$ functions of the fundamental set of solutions of the system, C_{nqk} are arbitrary constants.

From (12), (15) the representation of the harmonic function in a local domain follows

$$F(\rho_1,\theta_1,s_1) = \rho_1^{m_0}\Big(\sum_{n=0}^{K}(D_{n1}(\theta_1)\cos nls_1 + D_{n2}(\theta_1)\sin nls_1) + O(1/K)\Big) + O(\rho_1^{m_0+1}). \tag{16}$$

If $m_0 \in (0,1)$, then the normal derivative takes singularities of exponential character at the conic point

$$\partial F/\partial n = \rho_1^{m_0-1}\Big(\sum_{n=0}^{K}(dD_{n1}(\theta_1)/d\theta_1\cos nls_1 + dD_{n2}(\theta_1)/d\theta_1\sin nls_1) + O(1/K)\Big) + O(\rho_1^{m_0}). \tag{17}$$

Assume that H_{01} in the Lame coefficients of relation (11) does not depend on the variable s_1, which corresponds, for example, to circular cone. Then we have $H_{01} = (1+d^2)^{1/2}\sin\theta_{11}$, $\theta_{11} = \theta_1+\beta$, $\operatorname{tg}\beta = d$. In addition, the function $D(\theta_1)$ in representation (12) is a function of one variable θ_1, and equation (13) passes into the Legendre equation (see [9], p. 125) and the harmonic function can be represented as follows

$$F(\theta_1) = C_0 \rho_1^{m_0} P_{m_0}(\cos\theta_1) + O(\rho_1^{m_0+1}), \tag{18}$$

while its normal derivative for $m_0 \in (0,1)$ takes a singularity at the conic point

$$\partial F/\partial n = \rho_1^{m_0-1} C_0 dP_{m_0}(\cos\theta_1)/d\theta_1 + O(\rho_1^{m_0}), \tag{19}$$

where $P_{m_0}(\cos\theta_1)$ are the Legendre functions, C_0 is a constant.

Let us note that into representations (18), (19) only Legendre functions of the first genus entered being bounded for $\theta_1 \in [-\pi, \pi)$.

By substituting expressions for harmonic functions and their derivatives of the type (16), (17) into conditions (3) written in the local coordinates ρ_1, θ_1, s_1, assuming $\theta_1 = \pi/2$, and equaling to zero the expressions at same powers of ρ_1 and harmonics, we obtain a homogeneous system of linear algebraic equations with respect to arbitrary constants. By equaling the determinant of this system to zero, we obtain the characteristic equation for the determination of the value m_0

$$\Delta_0(m_0) = 0. \tag{20}$$

Treating in the same manner representations (18), (19), we find the characteristic equation

$$P_{m_0}(\cos\theta_1) = 0. \tag{21}$$

Thus, we have proved the following

Lemma 2. *If characteristic equations* (20), (21) *possess the roots* $m_0 \in (0,1)$, *then the harmonic functions and their normal derivatives possess the asymptotic* (16)–(19).

Let us note that the set of roots $m_0 \in (0,1)$ of the characteristic equations (20), (21) is nonempty (see [10], p. 320).

3. *Solvability of the problem.* Let us pass to the general case of the problem of conjugation (1)–(3).

Theorem. *If at every point of a singular line the condition is fulfilled*: $m = \pi/(2\pi - \varpi) < 1$, *and at every conic point the roots* $m_0 \in (0,1)$ *of characteristic equations* (20), (21) *exist, then the problem of conjugation* (1)–(3) *is unconditionally solvable.*

Proof. First we consider the case of one finite domain V_1 with the boundary surface $S = \bigcup_{j=1}^{n_1} S_j$ containing the conic points O_k, $k = \overline{1, m_1}$, and smooth closed non-intersecting singular lines $L_j = S_j \cap S_{j+1}$, $j = \overline{1, n_1 - 1}$, $O_k \notin L_j$. The surfaces S_j are defined by the equations $f_j(x,y,z) = 0$, while the surfaces expanded over the contours L_j are described by the equations $f_{0j}(x,y,z) = 0$.

We represent the harmonic functions in the form

$$F_0(M) = F_{01}(M) + F_{02}(M), \quad M \in V_0, \tag{22}$$

$$F_1(M) = F_{11}(M) + F_{12}(M), \quad M \in V_1, \tag{23}$$

where the first addends realize the asymptotic (6), (16)–(19) which depends on the type of non-smoothness, while the second addends are bounded continuous functions in their domains of definition.

The set of points of smoothness of the surface S is a piecewise smooth surface. Let us represent the first addends by means of potentials of either simple or double layers, which are harmonic functions vanishing at infinity (see [11], p. 398),

$$F_{i1}(M) = \int_S \Big(\prod_{j=1}^{n_1-1} \mu_{ij1}^{(1)}(N)\Big)\Big(\prod_{k=1}^{p_1} \mu_{ik1}^{(2)}(N)\Big) R^{-1} ds + \\ + \iint_S \Big(\prod_{j=1}^{n_1-1} \mu_{ij1}^{(3)}(N)\Big)\Big(\prod_{k=1}^{p_1} \mu_{ik1}^{(4)}(N)\Big) \partial R^{-1}/\partial n\, ds, \quad (24)$$

where $N = N(\xi,\eta,\chi) \in S$, $R = ((x-\xi)^2 + (y-\eta)^2 + (z-\chi)^2)^{1/2}$, $i = 0, 1$, the derivatives are taken along the direction of the exterior normal to S.

For singular lines we take the densities in the form

$$\mu_{ij1}^{(3)}(N) = (-1)^{i+1}/(2\pi) R_j^{m_j}(N)(C_{i1j}\cos(m_j T_j(N)) + C_{i2j}\sin(m_j T_j(N))), \quad (25)$$

$$\mu_{ij1}^{(1)}(N) = (-1)^{i+1} m_j/(2\pi) R_j^{m_j-1}(N)(C_{i2j}\cos(m_j T_j(N)) - C_{i1j}\sin(m_j T_j(N))), \quad (26)$$

where

$$R_j(N) = ((f_{0j}(N)/|\operatorname{grad} f_{0j}(N)|)^2 + Q_j^2(N))^{1/2},$$
$$T_j(N) = \arcsin(f_{0j}(N)/(|\operatorname{grad} f_{0j}(N)| R_j(N))),$$
$$Q_j(N) = f_j(N)/|\operatorname{grad} f_j(N)| + (1 - (\mathbf{G}_{2j}(N), \mathbf{G}_{0j}(N))^2 f_{0j}(N)/|\operatorname{grad} f_{0j}(N)|)(\mathbf{G}_{0j}(N), \mathbf{G}_{2j}(N)),$$
$$\mathbf{G}_j(N) = \operatorname{grad} f_j(N)/|\operatorname{grad} f_j(N)|,$$
$$\mathbf{G}_{0j}(N) = \operatorname{grad} f_{0i}(N)/|\operatorname{grad} f_{0j}(N)|,$$
$$\mathbf{G}_{2j}(N) = [[\mathbf{G}_j(N), \mathbf{G}_{j+1}(N)]/|[\mathbf{G}_j(N), \mathbf{G}_{j+1}(N)]|, \mathbf{G}_{0j}(N)].$$

For conic points we have

$$\mu_{ik1}^{(4)}(N) = (-1)^{i+1}/(2\pi) R_k^{m_k}(N) \sum_{n=0}^{K} (D_{ikn1}(T_k(N))\cos(n\pi S_k(N)/s_{k0}) + \\ + D_{ikn2}(T_k(N))\sin(n\pi S_k(N)/s_{k0})), \quad (27)$$

$$\mu_{ik1}^{(2)}(N) = (-1)^{i+1}/(2\pi) R_k^{m_k-1}(N) \sum_{n=0}^{K} (dD_{ikn1}(T_k(N))/dT_k(N)\cos(n\pi S_k(N)/s_{k0}) + \\ + dD_{ikn2}(T_k(N))/dT_k(N)\sin(n\pi S_k(N)/s_{k0})), \quad (28)$$

where

$$R_k(N) = ((\xi - x_k)^2 + (\eta - y_k)^2 + (\chi - z_k)^2)^{1/2},$$
$$T_k(N) = \arcsin(l_{0k}(N)(\xi - x_k) + m_{0k}(N)(\eta - y_k) + n_{0k}(N)(\chi - z_k)),$$
$$S_k(N) = \int_0^{\varphi_k} ((dl_{0k}(N)/d\varphi_k)^2 + (dm_{0k}(N)/d\varphi_k)^2 + (dn_{0k}(N)/d\varphi_k)^2)^{1/2} d\varphi_k,$$

$\varphi_k = \varphi_k(\xi,\eta)$, $O_k = O_k(x_k, y_k, x_k)$, $l_{0k}(N)$, $m_{0k}(N)$, $n_{0k}(N)$ are coordinates of the unit vector of the generatrix $\mathbf{r}_{0k}(N)$, D_{iknq} are known functions in representation (15).

Similar formulas in the particular case of the asymptotic (18), (19) are

$$\mu_{ik1}^{(4)}(N) = (-1)^{i+1}/(2\pi) R_k^{m_k}(N) C_{0k} P_{m_k}(\cos T_{1k}(N)),$$
$$\mu_{ik1}^{(2)}(N) = (-1)^{i+1}/(2\pi) R_k^{m_k-1}(N) C_{0k}\, dP_{m_k}(\cos T_{1k}(N))/dT_{1k}(N),$$

where $T_{1k}(N) = T_k(N) + \beta$, P_{m_k} are the Legendre functions.

Substituting harmonic functions (22), (23) with regard for (24)–(28) into boundary conditions (3), we verify that they are satisfied and both the functions and their derivatives possess the asymptotic (6), (16)–(19). In addition, we take into account that by Lemmas 1, 2 we have $m_j, m_k \in (0,1)$ and the boundary values of the potentials of either simple or double layers at the smoothness points of the conjugation surface are defined by known values (see [11]), and we pass in them to the local coordinates in accordance with (7), (11).

The second addends of harmonic functions (22), (23) are written by potentials of simple and double layers with the unknown densities $\mu_{i2q}(N)$, $q = 1, 2$,

$$F_{02}(M) = F_{00}(M) + \iint_S \mu_{021}(N) R^{-1} ds + \iint_S \mu_{022}(N) \partial R^{-1}/\partial n \, ds, \tag{29}$$

$$F_{12}(M) = F^*_{11}(M) + \iint_S \mu_{121}(N) R^{-1} ds + \iint_S \mu_{122}(N) \partial R^{-1}/\partial n \, ds, \tag{30}$$

where $F_{00}(M)$, $F^*_{11}(M)$ are given harmonic addends stipulated by the physical content of the problem (see [3]).

To determine the densities we substitute (22), (23), (29), (30) into boundary conditions (1), (2) and assume in view of arbitrarinesses of the densities

$$\mu_{121}(N) = \mu_{021}(N), \quad \mu_{122}(N) = \mu_{022}(N).$$

As a result, we find

$$\mu_{022}(N) = h_0(N),$$

and for the density $\mu_{021}(N)$ we obtain the integral equation

$$\mu_{021}(N_0) - (\sigma/(2\pi))\partial\Big(\iint_S \mu_{021}(N) R^{-1} ds\Big)/\partial n = g_0(N_0), \tag{31}$$

where $N_0 \in S$,

$$h_0(N) = (F^-_{01}(N) + F^+_{00}(N) - (F^+_{11}(N) + F^{*+}_{11}(N)))/(4\pi),$$

$$g_0(N_0) = (\gamma(\partial F^-_{01}/\partial n + \partial F^-_{00}/\partial n) - (\partial F^+_{11}/\partial n + \partial F^{*+}_{11}/\partial n))/(2\pi(1+\gamma)) -$$

$$-(\sigma/(2\pi))\partial\Big(\iint_S h_0(N) \partial R^{-1}/\partial n \, ds\Big)/\partial n,$$

$$F_{i1} = F_{i1}(N_0), \quad F_{00} = F_{00}(N_0), \quad F^*_{11} = F^*_{11}(N_0), \quad \sigma = (1-\gamma)/(1+\gamma), \quad \gamma = \lambda_0/\lambda_1.$$

The right-hand side of equation (31) in view of the fulfillment of conditions (3) is a continuous bounded function on S, therefore this is an equation with polar kernel (see [11], p. 279).

The homogeneous associated equation

$$\mu^*_{021}(N_0) - (\sigma/(2\pi)) \iint_S \mu^*_{021}(N) \partial R^{-1}/\partial n \, ds = 0$$

can be treated as the fulfillment of condition (1) for the function

$$F^*(M) = \Gamma \iint_S \mu^*_{021}(N) \partial R^{-1}/\partial n \, ds;$$

besides, $\Gamma = \gamma$ if $M \in V_0$, and $\Gamma = 1$ if $M \in V_1$. In view of (1) it is a continuous harmonic function in the whole space, which takes a constant value on S in view of (2) and identically vanishes at infinity. Consequently, this function is identically equal to zero (see [11], p. 374).

In the case of several non-intersecting finite simply connected domains V_i, $i = \overline{1, k_0}$, the harmonic functions $F_i(M)$, $M \in V_i$, are to be constructed as in (23), (24) for every domain. A harmonic function $F_0(M)$, $M \in V_0$, can be taken in the form (24), where

$$F_{01}(M) = \sum_{i=1}^{k_0} F_{0i1}(M), \quad F_{02}(M) = \sum_{i=1}^{k_0} F_{0i2}(M);$$

here the addends $F_{0i1}(M)$ are constructed with regard for non-smoothness of the corresponding surfaces S_i as in (24)–(28), while the addends $F_{0i2}(M)$ can be represented in the form (29). □

For concrete physical problems the problem considered above can be complemented by conditions analogous to the two-dimensional case (see [12], [13]).

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Lutsk State Technical
University (Ukraine)