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The Closeness and Closability Criteria for Infinitesimal Operators of Certain Semigroups

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Abstract—We consider generators of certain operator semigroups and study the closeness and closability properties of certain generators (in the operator class). We obtain the necessary and sufficient conditions for these properties.

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1. INTRODUCTION

Let X be a complex Banach space and let the symbol $\operatorname{End} X$ denote the Banach algebra of endomorphisms (linear bounded operators) of the Banach space X.

Definition 1.1. We understand an *operator semigroup* as a strongly continuous operator-valued function $T : (0, \infty) \to \operatorname{End} X$ such that $T(t+s) = T(t)T(s)$ for any $t, s > 0$.

According to [1] (P. 316), we introduce the following definition.

Definition 1.2. An *infinitesimal operator* of a semigroup T is a linear operator $A_0 : D(A_0) \subset X \to X$ such that

$$A_0 x_0 = \lim_{t \to 0+} \frac{T(t)x_0 - x_0}{t}, \;\; x_0 \in D(A_0) = \left\{ x \in X : \text{ there exists } \lim_{t \to 0+} \frac{T(t)x - x}{t} \right\}.$$

In this paper we impose no usual restrictions on the operator semigroup T. Namely, we do not require that the function $T : (0, \infty) \to \operatorname{End} X$ should be strongly continuous at zero. Moreover, we do not require that the kernel $\operatorname{Ker} T = \{x \in X : T(t)x = 0 \text{ for all } t > 0\}$ of the semigroup T should satisfy the condition $\operatorname{Ker} T = \{0\}$. The notion of a *generator* of an operator semigroup was introduced for such general semigroups in the paper [2] (P. 179). In this paper we study conditions for the closeness and non-closability (in the operator class) of some generators of the mentioned semigroups. The non-closability property means that the closure of an operator is a linear relation, but is not an operator.

2. THE MAIN NOTIONS OF THE THEORY OF LINEAR RELATIONS AND OPERATOR SEMIGROUPS

In what follows the symbol X stands for a Banach space, and $\operatorname{End} X$ does for the Banach algebra of its endomorphisms. Let us give some definitions ([3], pp. 3–5) of the theory of linear relations.

Definition 2.1. Any linear subspace $\mathcal{A}$ in the Cartesian product $X \times X$ is called a *linear relation* on the Banach space X. A linear relation is called *closed* if $\mathcal{A}$ is a closed linear subspace in $X \times X$.

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