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A Realization of Cartan Extensions

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Abstract—We revise the notion of a Cartan extension and consider in detail simple examples of known Cartan extensions.

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1. Basic definitions. A grading of a Lie algebra $L = \bigoplus_{k=-\infty}^{\infty} L_k$ over an arbitrary field $\mathbb{K}$ is called *transitive*, if the subalgebra $L^- = \bigoplus_{k=-\infty}^{-1} L_k$ is generated (as a Lie algebra) by the homogeneous component L_{-1} and $[L_{-1}, a] \neq 0$ for every nonzero element $a \in L^+ = \bigoplus_{k=0}^{\infty} L_k$. An algebra L will be called *transitive* if the grading under consideration is clear from the context. In particular, a Lie algebra $L^- = \bigoplus_{k=-\infty}^{-1} L_k$ is transitive if it is generated by L_{-1}.

Let L be a Lie algebra with transitive grading. A maximal Lie algebra $R = \bigoplus_{k=-\infty}^{\infty} R_k$ with transitive grading such that $R_k = L_k$ for all $k < 0$ and $R_k \supseteq L_k$ for all $k \geqslant 0$ is called a *Cartan extension* of L.

Obviously, if L and L' are two transitive Lie algebras whose negative parts coincide (i.e., $L^- = L'^-$), then their Cartan extensions also coincide. In particular, this takes place for Cartan extensions of L and its negative part, i.e., the subalgebra L^-.

A Cartan extension can be defined inductively. By virtue of the above remark, we let $L = L^-$.

Let $L = \bigoplus_{i=-d}^{-1} L_i$ be a transitive Lie algebra over a field $\mathbb{K}$. Consider the graded Lie algebra $R = \bigoplus_{i=-d}^{\infty} R_i$ over the same field $\mathbb{K}$ defined by the following conditions:

1) $R_i = L_i$ for $i = -d, \dots, -1$ (i.e., one can identify $R^- = L^-$);

2) R_0 is the set of all homogeneous derivations h of L (i.e., such derivations that $L_i h \subseteq L_i$, $i = -d, \dots, -1$, and $[x, y]h = [xh, y] + [x, yh] \quad \forall\, x, y \in L$);

3) for $i > 0$, R_i is the set of all differential operators f mapping the algebra L^- into $R = \bigoplus_{j=i-d}^{i-1} R_j$ and such that $L_j f \subseteq R_{j+i}$ for $j = -d, \dots, -1$.

The Lie multiplication on $R = \bigoplus_{i=-d}^{\infty} L_i$ is also defined inductively: For $f \in R_i$, $g \in R_j$, and $x \in L = R^-$, the element $[f, g] \in R_{i+j}$ is defined by $x[f, g] = [xf, g] + [f, xg]$ (since a homogeneous element is uniquely defined by its action on L).