УЧЕНЫЕ ЗАПИСКИ КАЗАНСКОГО УНИВЕРСИТЕТА.
СЕРИЯ ФИЗИКО-МАТЕМАТИЧЕСКИЕ НАУКИ

2018, Т. 160, кн. 2
С. 275–286

ISSN 2541-7746 (Print)
ISSN 2500-2198 (Online)

UDK 517.986

COVERINGS OF SOLENOIDS AND AUTOMORPHISMS OF SEMIGROUP C^*-ALGEBRAS

R.N. Gumerov

Kazan Federal University, Kazan, 420008 Russia

Abstract

The paper deals with finite-sheeted covering mappings onto the P-adic solenoids and limit endomorphisms of semigroup C^*-algebras. The aim of our exposition is two-fold: firstly, to present the results concerning the above-mentioned mappings and endomorphisms; secondly, to demonstrate proofs for some of the results. It has been shown that every covering mapping onto a solenoid is isomorphic to a power mapping. We have considered dynamical properties of the covering mappings. A power mapping for the P-adic solenoid is topologically transitive. A criterion for the covering mapping to be chaotic has been given. The classical Euler–Fermat theorem may be used in its proof. We have studied limit endomorphisms of C^*-algebras generated by isometric representations for semigroups of rational numbers. We formulate criteria for limit endomorphisms to be automorphisms in number-theoretic, algebraic, and functional terms. The necessity of such a criterion has been given from the category-theoretic viewpoint.

Keywords: automorphism of C^*-algebras, chaotic, inductive sequence of Toeplitz algebras associated with sequence of prime numbers, inverse limit and sequence, finite-sheeted covering mapping, semigroup C^*-algebra, solenoid, $*$-homomorphism, Toeplitz algebra, topologically transitive

Introduction

Throughout the paper, $P = (p_1, p_2, \ldots)$ is an arbitrary sequence of prime numbers. The P-*adic solenoid* Σ_P is the inverse limit of the inverse sequence

$$\mathbb{S}^1 \xleftarrow{f_1} \mathbb{S}^1 \xleftarrow{f_2} \mathbb{S}^1 \xleftarrow{f_3} \cdots,$$

where $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$, and $f_n(z) = z^{p_n}$, $z \in \mathbb{S}^1$, $n \in \mathbb{N}$. Thus, Σ_P is a subspace consisting of threads in the Cartesian product $\prod\limits_{n\in\mathbb{N}} \mathbb{S}^1$ endowed with the Tichonoff topology:

$$\Sigma_P = \varprojlim\{\mathbb{S}^1, f_n\} = \{(z_1, z_2, \ldots) : z_n \in \mathbb{S}^1, z_{n+1}^{p_n} = z_n, n \in \mathbb{N}\}.$$

If P is the constant sequence $(2, 2, \ldots)$, then the solenoid Σ_P is said to be *dyadic*.

P-adic solenoids are commutative topological groups. Two sequences of prime numbers P and Q are said to be *equivalent* if a finite number of terms can be deleted from each sequence, so that every prime number occurs the same number of times in the deleted sequences. It is known that the topological groups Σ_P and Σ_Q are isomorphic if and only if the sequences P and Q are equivalent.

P-adic solenoids were introduced by L. Vietoris in 1927 [1] and D. van Dantzig and B. van der Waerden in 1928 [2] (see also [3]). These objects have been studied for a long time (see the references below and the literature therein). Solenoids have been the focus of attention in various branches of mathematics and physics. In particular, they provide topological models for the Smale attractors.

As is well known, properties of objects and morphisms in the algebraic and topological categories have appropriate analogs for objects and morphisms in the categories of Banach algebras. P-adic solenoids are closely related to semigroup C^*-algebras.

The motivation for this work comes from the sources of two kinds. On the one hand, these are results on covering mappings onto topological groups. On the other hand, these are papers on semigroup C^*-algebras. The corresponding references will be given in the text. Here, we mention only Pontryagin's theorem [4, theorem 79] on the existence of a group structure on a covering space of a connected locally path connected topological group making a given covering mapping into a morphism of topological groups.

Pontryagin's theorem. *Let $f : X \to G$ be a covering mapping from a path connected space X onto a connected locally path connected topological group G with identity e. Then, for any point $\widetilde{e} \in f^{-1}(e)$, there exists a unique structure of the topological group on X, such that $\widetilde{e}$ is the identity and $f : X \to G$ is a morphism of topological groups. Furthermore, if G is abelian, then f is a morphism of abelian groups.*

Pontryagin's theorem is one of starting points of our research.

This paper is concerned with finite-sheeted covering mappings onto the P-adic solenoids, as well as endomorphisms of semigroup C^*-algebras for semigroups of rational numbers. We treat C^*-algebras as the inductive limits of inductive sequences of Toeplitz algebras associated with sequences of prime numbers. It is worth noting that the semigroup C^*-algebra is a natural object, because it is generated by the left regular representation for an additive semigroup of non-negative rational numbers. The endomorphism is considered as the limit $*$-homomorphism induced by a morphism between two copies of the same inductive sequence of Toeplitz algebras.

1. Preliminaries

In this section we consider finite-sheeted covering mappings from topological spaces onto topological groups. All spaces are assumed to be Hausdorff.

Definition 1. For $k \in \mathbb{N}$, a surjective mapping $f : X \to Y$ between topological spaces X and Y is called a *k-sheeted* (*k-fold*) *covering mapping* provided that for each $y \in Y$ there exists an open neighborhood $W \subset Y$ of y, such that the inverse image $f^{-1}(W)$ can be written as the disjoint union of k open sets lying in X, each of which is mapped homeomorphically onto W under f. If X is a connected space, then $f : X \to Y$ is said to be *a connected covering.*

Now, we recall the definition of isomorphic covering mappings.

Definition 2. A finite-covering mapping $f_1 : X_1 \to Y$ is said to be *isomorphic* (or *equivalent*) to a covering $f_2 : X_2 \to Y$ provided that there exists a homeomorphism $\rho : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[rr, "\rho"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

i.e., the equality $f_1 = f_2 \circ \rho$ holds.

Example 1. Each homeomorphism is an 1-sheeted covering mapping.

Example 2. For a topological space X and $k \in \mathbb{N}$, the projection onto the first coordinate $f : X \times \{1, 2, \ldots, k\} \to X : (x, m) \mapsto x, \quad m = 1, \ldots, k,$ is a k-sheeted covering mapping. We recall that a k-sheeted covering mapping onto X is said to be *trivial* if it is equivalent to that projection.

Example 3. The k-th power mapping $f_k : \mathbb{S}^1 \to \mathbb{S}^1 : z \mapsto z^k$ is a k-sheeted covering mapping.

The P-adic solenoid is a compact connected abelian group under the coordinatewise multiplication with the identity $(1, 1, \ldots)$.

One of the main tools in studying covering mappings onto topological groups is Pontryagin's theorem formulated in Introduction. But the P-adic solenoid is not locally connected at any point. To study the structure of finite-sheeted covering mappings on topological groups which are not locally connected, we make use of the following statement [5–7].

Theorem 1. *Let $f : X \to G$ be a finite-sheeted covering mapping from a connected space X onto a compact group G with identity e. Then, for any point $\widetilde{e} \in f^{-1}(e)$, there exists a unique structure of the topological group on X such that $\widetilde{e}$ is the identity and $f : X \to G$ is a continuous homomorphism between the compact groups. Furthermore, if G is abelian, then f is a continuous homomorphism of the abelian groups.*

It follows from this result that all finite-sheeted covering mappings onto the compact connected abelian groups are defined, up to isomorphisms, by polynomials with continuous coefficients. There are also applications of this theorem to trivial coverings and to algebraic equations with functional coefficients [5–9]. To present here some of those applications, we give necessary notations and definitions.

In what follows, the symbol G stands for an additive compact connected abelian group and $\widehat{G}$ denotes its additive character group, i.e., the group of continuous homomorphisms from G to the multiplicative group $\mathbb{S}^1$.

For the given $k \in \mathbb{N}$ we consider the homomorphism of discrete groups defined by

$$\tau_k : \widehat{G} \to \widehat{G} : \chi \mapsto k\chi.$$

Definition 3. Let $k \in \mathbb{N}$. We say that the group $\widehat{G}$ is *k-divisible* provided that the homomorphism τ_k is surjective.

Note that if $\widehat{G}$ is k-divisible, then the homomorphism τ_k is an automorphism, because $\widehat{G}$ is a torsion-free abelian group [10, theorem 24.25].

As a corollary of theorem 1, one can obtain the following criterion for triviality of covering mappings onto compact abelian groups.

Theorem 2. *Let $k \geqslant 2$ be an integer. Then all k-sheeted covering mappings onto a compact connected abelian group G are trivial if and only if the character group $\widehat{G}$ is $k!$-divisible.*

It is worth noting that the necessary and sufficient conditions for $\widehat{G}$ to be k-divisible are given in [11]. In particular, it was shown there that the character group $\widehat{G}$ is k-divisible if and only if the group G admits a k-mean. Below we recall the definition of a mean.

Definition 4. For an integer $k \geqslant 2$, a *k-mean* on G is a continuous mapping

$$\mu : G \times G \times \cdots \times G \to G$$

from the Cartesian product of k copies of G satisfying the following conditions:

1) $\mu(g, g, \ldots, g) = g$;
2) $\mu(g_1, g_2, \ldots, g_k) = \mu(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(k)})$
for all elements $g, g_1, g_2, \ldots, g_k \in G$ and for every permutation π of the set $\{1, 2, \ldots, k\}$.

The question on the existence of means on solenoids was considered in [12, 13].

Finally, we note that various generalizations of Pontryagin's theorem are provided in [14–17].

2. Coverings of P-adic solenoids

In what follows, we shall consider finite-sheeted covering mappings onto P-adic solenoids. There is a vast literature on these coverings. Note that R. Fox's theory of overlays was applied to the study of coverings onto the solenoids in 1972 (see [18–20]).

Let us take $k \in \mathbb{N}$. We consider the morphism $\{h_n^k : n \in \mathbb{N}\}$ between two copies of the same inverse sequence and the limit mapping h_P^k induced by this morphism:

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[d, "h_1^k"'] & \mathbb{S}^1 \arrow[l, "f_1"'] \arrow[d, "h_2^k"] & \mathbb{S}^1 \arrow[l, "f_2"'] \arrow[d, "h_3^k"] & \arrow[l, "f_3"'] \cdots & \Sigma_P \arrow[d, "h_P^k"] \\
\mathbb{S}^1 & \mathbb{S}^1 \arrow[l, "f_1"'] & \mathbb{S}^1 \arrow[l, "f_2"'] & \arrow[l, "f_3"'] \cdots & \Sigma_P,
\end{tikzcd}$$

where $h_n^k(z) = z^k$ and $z \in \mathbb{S}^1$. Therefore, we get $h_P^k(g) = g^k$, where $g \in \Sigma_P$.

Various properties of the mappings $h_P^k : \Sigma_P \to \Sigma_P$ are considered in [21, 22, 24]. It is worth noting that the paper [23] is devoted to so-called generalized solenoids.

It is known that every mapping $h_P^k : \Sigma_P \to \Sigma_P$ is a finite-sheeted covering (see, for example, [21, lemma 1], [22, proposition 8], [24, proposition 1]). More precisely, we have the following assertions concerning those mappings ([24, proposition 3, theorem 1 and theorem 2]).

Theorem 3. *Let P be a sequence of prime numbers and let $k \geqslant 2$ be an integer. Then, the limit mapping $h_P^k : \Sigma_P \to \Sigma_P$ is a homeomorphism if and only if every prime divisor of k is equal to infinitely many terms of the sequence P.*

Theorem 4. *Let P be a sequence of prime numbers and let $k \geqslant 2$ be an integer. Then, the mapping h_P^k is a k-sheeted connected covering of the P-adic solenoid Σ_P if and only if every prime divisor of k is equal to a finite number of terms in the sequence P.*

It is interesting to note that theorem 3 and theorem 4 are applied to the crossed product C^*-algebras in [25].

Theorem 5. *Let $f : X \to \Sigma_P$ be a k-fold connected covering of the P-adic solenoid Σ_P. Then, f is is isomorphic to h_P^k.*

Proof. We shall give a proof of the theorem by making use of the results in [5–7] (see also [24, remark 2]). The approximation construction given there yields the following commutative diagram:

$$\begin{tikzcd}
(X_n, e_n) \arrow[dr, "\eta_n"] & & & (X_{n+1}, e_{n+1}) \arrow[lll, "\zeta_n"'] \arrow[dl, "\eta_{n+1}"'] \\
& (\mathbb{S}^1, 1) & (\mathbb{S}^1, 1) \arrow[l, "f_n"'] & \\
(\mathbb{S}^1, 1) \arrow[uu, dotted, "\phi_n"] \arrow[ur, "h_n^k"'] & & & (\mathbb{S}^1, 1) \arrow[lll, "f_n"] \arrow[ul, "h_{n+1}^k"] \arrow[uu, dotted, "\phi_{n+1}"']
\end{tikzcd}$$

Here, firstly, the family $\{(X_n, e_n), \zeta_n\}$ is an inverse sequence consisting of connected locally path, connected pointed spaces, and continuous mappings of pointed spaces. Secondly, for every $n \in \mathbb{N}$, there exists a continuous surjection $\tau_n : X \to X_n$, such that $\tau_n = \zeta_n \circ \zeta_{n+1} \circ \ldots \circ \zeta_m \circ \tau_m$ whenever $n < m$. Thirdly, the sequence $\{\eta_n : (X_n, e_n) \to (\mathbb{S}^1, 1)\}$ is a morphism between inverse sequences $\{(X_n, e_n), \zeta_n\}$ and $\{(\mathbb{S}^1, 1), f_n\}$. Fourthly, every η_n is a k-sheeted covering mapping of pointed spaces.

Let the morphism $\{\eta_n : (X_n, e_n) \to (\mathbb{S}^1, 1)\}$ induce the limit mapping

$$\varprojlim\{\eta_n\} : \varprojlim\{(X_n, e_n), \zeta_n\} \longrightarrow \Sigma_P.$$

Applying the universal property of the inverse limit $\varprojlim\{(X_n, e_n), \zeta_n\}$ to the sequence $\{\tau_n\}$, we get the continuous mapping $\varrho : X \longrightarrow \varprojlim\{(X_n, e_n), \zeta_n\}$. It is straightforward to check that ϱ is a homeomorphism. Moreover, it establishes an isomorphism between f and $\varprojlim\{\eta_n\}$ (see [6, p. 10]).

It follows from the well-known lifting lemmas in the classical theory of covering mappings that every mapping h_n^k can be lifted with respect to the covering mapping $\eta_n : (X_n, e_n) \longrightarrow (\mathbb{S}^1, 1)$ to a homeomorphism of pointed spaces

$$\phi_n : (\mathbb{S}^1, 1) \longrightarrow (X_n, e_n).$$

The commutativity of the above diagram yields the following equalities:

$$\eta_n \circ \phi_n \circ f_n = h_n^k \circ f_n = f_n \circ h_{n+1}^k = f_n \circ \eta_{n+1} \circ \phi_{n+1} = \eta_n \circ \zeta_n \circ \phi_{n+1}.$$

By the uniqueness of liftings, we see that the equality $\phi_n \circ f_n = \zeta_n \circ \phi_{n+1}$ holds for each $n \in \mathbb{N}$. Therefore, the sequence of homeomorphisms $\{\phi_n\}$ is a morphism between the inverse sequences $\{(\mathbb{S}^1, 1), f_n\}$ and $\{(X_n, e_n), \zeta_n\}$. Consequently, we can consider the limit mapping, which is a homeomorphism:

$$\varprojlim\{\phi_n\} : \Sigma_P \longrightarrow \varprojlim\{(X_n, e_n), \zeta_n\}.$$

In addition, we have the equality: $\varprojlim\{\eta_n\} \circ \varprojlim\{\phi_n\} = h_P^k$. Thus, we conclude that the mappings $\varprojlim\{\eta_n\}$ and h_n^k, and, hence, f and h_n^k are isomorphic, as required. □

Further, we discuss dynamical properties of the finite-sheeted covering mappings onto the P-adic solenoids (see also [21, 22, 24, 26]).

Definition 5. For $n \in \mathbb{N}$, the composite $h_P^k \circ h_P^k \circ \ldots \circ h_P^k$ (n times) is called *the n-th iteration* of h_P^k and is denoted by $(h_P^k)^n$ ($(h_P^k)^1 = h_P^k$). A point $x \in \Sigma_P$ is said to be *periodic* if there exists $n \in \mathbb{N}$, such that $(h_P^k)^n(x) = x$.

Definition 6. The covering mapping $h_P^k : \Sigma_P \to \Sigma_P$ is said to be *topologically transitive* provided that for each pair of open sets $U, V \subset \Sigma_P$ there exists $n \in \mathbb{N}$, such that $(h_P^k)^n(U) \cap V \neq \emptyset$.

Definition 7. A covering mapping $h_P^k : \Sigma_P \to \Sigma_P$ is said to be *chaotic* if it is topologically transitive and the set of its periodic points is dense in Σ_P.

The following theorem for an arbitrary P-adic solenoid is an analog of proposition 8 in [21] for the dyadic solenoid. Their proofs are similar.

Theorem 6. *A covering mapping $h_P^k : \Sigma_P \to \Sigma_P$ is topologically transitive for every integer $k \geqslant 2$.*

It is worth noting that we make use of the Euler–Fermat theorem in the proof of the following statement (see [24, proposition 6]).

Theorem 7. *Let P be a sequence of prime numbers and let k be an integer greater than 1. Then the covering mapping h_P^k is chaotic if and only if there exists a prime number q satisfying the following conditions:*

1) *q is not a divisor of the integer k;*
2) *$\exists N \in \mathbb{N}$, such that $q \neq p_n$ for all $n \geqslant N$.*

Now, we recall the classical result, which is usually referred as the Euler–Fermat theorem. Note that Fermat's little theorem is its special case.

The Euler–Fermat theorem. *Let a be a natural number coprime to $m \in \mathbb{N}$. Then*

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Here, $\varphi(m)$ is a value of the Euler function at m. That is, $\varphi(1) = 1$ and, for $m > 1$, the number $\varphi(m)$ is defined as follows:

$$\varphi(m) = q_1^{l_1-1} q_2^{l_2-1} \dots q_n^{l_n-1} (q_1 - 1)(q_2 - 1) \cdots (q_n - 1),$$

where $m = q_1^{l_1} q_2^{l_2} \dots q_n^{l_n}$ is the canonical factorization of m, so that, $q_1, q_2, \dots, q_n \geq 2$ are distinct prime numbers and $l_1, l_2, \dots, l_n \in \mathbb{N}$.

For the dynamics of generalized solenoids, we refer the reader to [23, § 6,9].

Finally, we note that there exists a connected covering space of the solenoid, which does not admit a topological group structure, such that a covering mapping becomes a morphism of topological groups. Of course, such a covering mapping is infinite-sheeted [27].

3. Limit endomorphisms of semigroup C^*-algebras

Together with the sequence $P = (p_1, p_2, \dots)$ we consider the dual group of the P-adic solenoid, i.e., the additive group of rational numbers:

$$\widehat{\Sigma_P} \simeq \mathbb{Q}_P := \Big\{ \frac{m}{p_1 p_2 \dots p_n} \Big| m \in \mathbb{Z}, n \in \mathbb{N} \Big\}.$$

Let Γ denote either the group $\mathbb{Z}$ or $\mathbb{Q}_P$. Let $\Gamma^+ = \Gamma \cap [0; +\infty)$ be a positive cone in the ordered group Γ. As usual, we denote by $l^2(\Gamma^+)$ the Hilbert space of all square summable complex-valued functions on the additive semigroup Γ^+. Let $\{e_a | a \in \Gamma^+\}$ be the canonical orthonormal basis in $l^2(\Gamma^+)$ given by $e_a(b) = \delta_{a,b}$, where $\delta_{a,b} = 1$ if $a = b$ and 0 if $a \neq b$.

Let $B(l^2(\Gamma^+))$ be the C^*-algebra of all bounded linear operators on the Hilbert space $l^2(\Gamma^+)$. For every $a \in \Gamma^+$, we define the isometry $V_a \in B(l^2(\Gamma^+))$ by setting

$$V_a e_b = e_{a+b},$$

where $b \in \Gamma^+$. The sub-C^*-algebra of $B(l^2(\Gamma^+))$ generated by the set $\{V_a | a \in \Gamma^+\}$ is called *the reduced semigroup C^*-algebra of* Γ^+, or *the Toeplitz algebra generated by* Γ^+. It is denoted by $C_r^*(\Gamma^+)$. In the case $\Gamma = \mathbb{Z}$, we denote it by $\mathcal{T}$ and use the symbols T and T^n instead of V_1 and V_n, respectively, where $n \in \mathbb{Z}^+$.

L.A. Coburn studied the Toeplitz algebra for the additive semigroup of non-negative integers [28]. R.G. Douglas [29] considered the case of subsemigroups in the additive group of all real numbers. G.J. Murphy [30] explored the general case of ordered groups. In the last decade, semigroup C^*-algebras have also attracted a considerable attention from researchers (see, for example, [31–34] and references therein).

Definition 8. *An isometric homomorphism(or representation)* from Γ^+ into a unital C^*-algebra B is a mapping given by

$$\rho : \Gamma^+ \to B : a \mapsto W_a$$

such that the equalities $W_a^* W_a = 1$ and $W_{a+b} = W_a W_b$ are valid for all $a,\, b \in \Gamma^+$.

Clearly, the mapping defined by the formula

$$\pi : \Gamma^+ \to C_r^*(\Gamma^+) : a \mapsto V_a$$

is an isometric homomorphism. Furthermore, it is the universal one. Explicitly, this means that π satisfies the following universal property (see [30, theorems 1.3,1.9]).

Theorem 8. *Let $\rho : \Gamma^+ \to B$ be an isometric homomorphism. Then, there exists a unique $*$-homomorphism $\rho^* : C_r^*(\Gamma^+) \to B$, such that the diagram*

$$\begin{tikzcd} C_r^*(\Gamma^+) \arrow[rr, "\rho^*"] & & B \\ & \Gamma^+ \arrow[ul, "\pi"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

is commutative, i.e., $\rho^* \circ \pi = \rho$. Moreover, if $\rho(a)$ is non-unitary for every $a > 0$, then the $*$-homomorphism ρ^* is injective.

We note that one can find self-contained proofs of some results from [30] in [35, corollaries 2.5, 2.6].

In the literature on the subject, the universal property of the isometric homomorphism for $\Gamma^+ = \mathbb{Z}^+$ is also known as the Coburn theorem, which is formulated in the following form (see, e.g., [36, theorem 3.5.18]).

Theorem 9. *Let V be an isometry in the unital C^*-algebra B. Then, there exists a unique unital $*$-homomorphism $\varphi : \mathcal{T} \to B$, such that $\varphi(T) = V$. Moreover, if $VV^* \neq 1$, then φ is isometric.*

In the sequel, for the $*$-homomorphism φ defined in the Coburn theorem, we shall write shortly $\varphi : \mathcal{T} \longrightarrow B : T \mapsto V$.

As an immediate consequence of the Coburn theorem, one has that for each positive integer n there exists a unique unital $*$-homomorphism of C^*-algebras

$$\varphi : \mathcal{T} \longrightarrow \mathcal{T} : T \mapsto T^n.$$

Furthermore, the $*$-homomorphism φ is isometric. Using such $*$-homomorphisms, we define inductive sequences of the Toeplitz algebras.

Definition 9. For an arbitrary sequence of prime numbers $P = \{p_1, p_2, \ldots\}$ and the sequence of isometric $*$-homomorphisms $\{\varphi_n : \mathcal{T} \longrightarrow \mathcal{T} : T \mapsto T^{p_n}\}_{n=1}^{\infty}$, the inductive sequence

$$\mathcal{T} \xrightarrow{\varphi_1} \mathcal{T} \xrightarrow{\varphi_2} \mathcal{T} \xrightarrow{\varphi_3} \ldots$$

is called *the inductive sequence of Toeplitz algebras associated with the sequence P*.

Theorem 10. *Let $P = \{p_1, p_2, \ldots\}$ be a sequence of prime numbers and let $\{\mathcal{T}, \varphi_n\}_{n=1}^{+\infty}$ be the inductive sequence of Toeplitz algebras associated with P. Then, there exists an isomorphism of C^*-algebras between the inductive limit of the given sequence of algebras and the Toeplitz algebra generated by the semigroup $\mathbb{Q}_P^+$:*

$$\varinjlim \{\mathcal{T}, \varphi_n\} \cong C_r^*(\mathbb{Q}_P^+).$$

This result can be considered as a partial case of theorem 1.6 from [30] about the continuity of the functor sending groups to Toeplitz algebras. The point is that the group $\mathbb{Q}_P$ is the inductive limit of the inductive sequence of groups:

$$\mathbb{Z} \xrightarrow{\tau_1} \mathbb{Z} \xrightarrow{\tau_2} \mathbb{Z} \xrightarrow{\tau_3} \ldots,$$

where the connecting homomorphisms are given by $\tau_n(m) = p_n m$, $m \in \mathbb{Z}$. One can check this well-known fact by making use of definitions.

For $k \in \mathbb{N}$, we consider the limit endomorphism $\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \to C_r^*(\mathbb{Q}_P^+)$ defined by the morphism $\{\varphi_n^k : n \in \mathbb{N}\}$ between two copies of the same inductive sequence of the Toeplitz algebras associated with P :

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\varphi_1"] \arrow[d, "\varphi_1^k"'] & \mathcal{T} \arrow[r, "\varphi_2"] \arrow[d, "\varphi_2^k"] & \mathcal{T} \arrow[r, "\varphi_3"] \arrow[d, "\varphi_3^k"] & \cdots & C_r^*(\mathbb{Q}_P^+) \arrow[d, "\varphi_P^k"] \\
\mathcal{T} \arrow[r, "\varphi_1"] & \mathcal{T} \arrow[r, "\varphi_2"] & \mathcal{T} \arrow[r, "\varphi_3"] & \cdots & C_r^*(\mathbb{Q}_P^+),
\end{tikzcd}$$

where the vertical morphisms are given by $\varphi_n^k : \mathcal{T} \longrightarrow \mathcal{T} : T \longmapsto T^k$.

It can be seen that the limit endomorphism $\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \to C_r^*(\mathbb{Q}_P^+)$ coincides with the $*$-homomorphism defined by the universal property of isometric homomorphisms:

$$\begin{tikzcd}
C_r^*(\mathbb{Q}_P^+) \arrow[rr, "\rho_k^*=\varphi_P^k"] & & C_r^*(\mathbb{Q}_P^+) \\
& \mathbb{Q}_P^+ \arrow[ul, "\pi"] \arrow[ur, "\rho_k"'] &
\end{tikzcd}$$

Here, the $*$-homomorphism ρ_k is given by the formula

$$\rho_k : \mathbb{Q}_P^+ \longrightarrow C_r^*(\mathbb{Q}_P^+) : a \longmapsto V_{ka}.$$

We have the following criterion for φ_P^k to be an automorphism of C^*-algebras.

Theorem 11. *Let P be a sequence of prime numbers and let $k \geqslant 2$ be an integer. Then, the limit endomorphism $\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is an automorphism if and only if each prime divisor of k is equal to infinitely many terms of the sequence P.*

For the complete proof of this theorem, we refer the reader to [37]. As noted in [37, remark 3.10], the proof of the “only if” part of theorem 11 can be established by a way different from that given in [37, section 3]. Below, we sketch out the way of the proof.

Proof. Let us assume that we are given a limit automorphism of C^*-algebras:

$$\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+).$$

Using results in [29, 30], we can consider the following commutative diagram:

$$\begin{tikzcd}
C_r^*(\mathbb{Q}_P^+) \arrow[r, "\varphi_P^k"] \arrow[d, "\psi"'] & C_r^*(\mathbb{Q}_P^+) \arrow[d, "\psi"] \\
C_r^*(\mathbb{Q}_P^+)/K \arrow[r, "\varphi"] \arrow[d, "\iota"'] & C_r^*(\mathbb{Q}_P^+)/K \arrow[d, "\iota"] \\
C(\widehat{\mathbb{Q}}_P) \arrow[r, "\widetilde{\varphi}"] & C(\widehat{\mathbb{Q}}_P)
\end{tikzcd}$$

Here, firstly, K is the commutator ideal of $C_r^*(\mathbb{Q}_P^+)$, i.e., the closed ideal generated by all elements of the form $ab - ba$, where $a, b \in C_r^*(\mathbb{Q}_P^+)$. It is clear that if $\xi : A \to B$ is a $*$-homomorphism of C^*-algebras, then $\xi(K(A)) \subseteq \xi(K(B))$, with the equality if ξ is surjective, where $K(A)$ and $K(B)$ are the commutator ideals of C^*-algebras A and B, respectively. Secondly, ψ is the quotient $*$-homomorphism, and φ, ι as well as $\widetilde{\varphi}$ are the isomorphisms of C^*-algebras. Thirdly, $\widehat{\mathbb{Q}}_P$ is the compact dual group of the discrete group $\mathbb{Q}_P$ and $C(\widehat{\mathbb{Q}}_P)$ is the commutative C^*-algebra of all complex-valued continuous functions on $\widehat{\mathbb{Q}}_P$.

Further, let us consider the spectrum functor (see, e.g., [38, Ch.IV, §1]):

$$\mathcal{UCBA} \longrightarrow \mathcal{CT}op,$$

i.e., the contravariant functor from the category of unital commutative Banach algebras and their continuous unital homomorphisms to the category of compact topological spaces and their continuous mappings. It sends the algebra $C(\widehat{\mathbb{Q}}_P)$ to its spectrum, i.e., the space of multiplicative functionals, or, equivalently, to the space of maximal ideals, which is homeomorphic to the compact space $\widehat{\mathbb{Q}}_P$. By the Pontryagin duality, the compact group $\widehat{\mathbb{Q}}_P$ is topologically isomorphic to the P-adic solenoid Σ_P.

One can see that the isomorphism $\widetilde{\varphi}$ really induces the k-th power mapping

$$h_P^k : \Sigma_P \longrightarrow \Sigma_P,$$

which is the isomorphism of topological groups.

By theorem 3, we conclude that each prime divisor of the number k occurs infinitely many times in the sequence P. This completes the proof. □

Using number-theoretic arguments, one can easily prove that, for a natural number $k \geqslant 2$, every prime divisor of k is equal to infinitely many terms in the sequence P if and only if the group of rational numbers Q_P is k-divisible. Combining this observation and the result from [11] on the existence of means (see section 2) with theorem 11, we obtain the following assertion.

Theorem 12. *Let P be a sequence of prime numbers and let $k \geqslant 2$ be an integer. The following conditions are equivalent:*

1) *the limit $*$-homomorphism $\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is an automorphism of C^*-algebras;*

2) *the group of rational numbers Q_P is k-divisible;*

3) *the P-adic solenoid Σ_P admits a k-mean.*

Finally, we consider several examples. Let $k \geqslant 2$ be an integer.

Example 4. Let $P = (2, 3, 5, 7, \ldots)$ be the sequence of all prime numbers. Then, the k-th power mapping $h_P^k : \Sigma_P \to \Sigma_P$ is a chaotic k-fold covering mapping. The limit endomorphism $\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is not an automorphism of C^*-algebras.

Example 5. We take a periodic sequence $P = (2, 3, 2, 3, 2, 3, \ldots)$. Then, the k-th power mapping $h_P^k : \Sigma_P \to \Sigma_P$ is a k-fold covering mapping if and only if neither 2 nor 3 is a divisor of k. Every mapping h_P^k is chaotic. An endomorphism $\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is an automorphism if and only if there exist non-negative integers m and n, such that $k = 2^m 3^n$.

Example 6. We consider the sequence $P = (2, 2, 3, 2, 3, 5, 2, 3, 5, 7, \ldots)$. We have the equality $\mathbb{Q}_P = \mathbb{Q}$, where $\mathbb{Q}$ is the group of all rational numbers. Then, the k-th power mapping $h_P^k : \Sigma_P \to \Sigma_P$ is a homeomorphism, which is not chaotic. The limit endomorphism $\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is an automorphism.

References

1. Vietoris L. Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen. *Math. Ann.*, 1927, vol. 97, pp. 454–472.
2. Dantzig van D., Waerden van der B.L. Über metrisch homogene Räume. *Abh. Math. Semin. Hamburg*, 1928, no. 6, pp. 367–376.
3. Dantzig van D. Über topologisch homogene Kontinua. *Fundam. Math.*, 1930, no. 15, pp. 102–125.
4. Pontryagin L.S. *Nepreryvnye gruppy* [Continuous Groups]. Moscow, Nauka, 1984. 520 p. (In Russian)
5. Grigorian S.A., Gumerov R.N. On a covering group theorem and its applications. *Lobachevskii J. Math.*, 2002, vol. 10, pp. 9–16.
6. Grigorian S.A., Gumerov R.N. On the structure of finite coverings of compact connected groups. *arXiv:math/0403329*, 2004. Available at: https://arxiv.org/pdf/math/0403329.pdf.
7. Grigorian S.A., Gumerov R.N. On the structure of finite coverings of compact connected groups. *Topol. Its Appl.*, 2006, vol. 153, no. 18, pp. 3598–3614. doi: 10.1016/j.topol.2006.03.010.
8. Gumerov R.N. Weierstrass polynomials and coverings of compact groups. *Sib. Math. J.*, 2013, vol. 54, no. 2, pp. 243–246. doi: 10.1134/S0037446613020080.
9. Gumerov R.N. Characters and coverings of compact groups. *Russ. Math.*, 2014, vol. 58, no. 4, pp. 7–13. doi: 10.3103/S1066369X14040021.
10. Hewitt E., Ross K.A. *Abstract Harmonic Analysis.* Vol. I: Structure of Topological Groups Integration Theory Group Representations. Berlin, Springer, 1963. 525 p. doi: 10.1007/978-1-4419-8638-2.
11. Keesling J. The group of homeomorphisms of a solenoid. *Trans. Amer. Math. Soc.*, 1972, vol. 172, pp. 119–131. doi: 10.2307/1996337.
12. Krupski P. Means on solenoids. *Proc. Am. Math. Soc.*, 2002, vol. 131, no. 6, pp. 1931–1933.
13. Gumerov R.N. On the existence of means on solenoids. *Lobachevskii J. Math.*, 2005, vol. 17, pp. 43–46.
14. Grigorian S.A. Gumerov R.N., Kazantsev A.V. Group structure in finite coverings of compact solenoidal groups. *Lobachevskii J. Math.*, 2000, vol. 4, pp. 39–46.
15. Eda K., Matijević V. Finite-sheeted covering maps over 2-dimensional connected, compact Abelian groups. *Topol. Its Appl.*, 2006, vol. 153, no. 7, pp. 1033-1045. doi: 10.1016/j.topol.2005.02.005.
16. Eda K., Matijevic V. Existence and uniqueness of group structures on covering spaces over groups. *Fundam. Math.*, 2017, vol. 238, pp. 241–267. doi: 10.4064/fm990-10-2016.
17. Dydak J. Overlays and group actions. *Topol. Its Appl.*, 2016, vol. 207, pp. 22–32.
18. Fox R.H. On shape. *Fundam. Math.*, 1972, vol. 74, pp. 47–71.
19. Fox R.H. Shape theory and covering spaces. In: Dickman R.F., Fletcher P. (Eds.) *Lecture Notes in Mathematics.* Berlin, Heidelberg, Springer 1974, vol. 375, pp. 71–90. doi: 10.1007/BFb0064013.
20. Moore T.T. On Fox's theory of overlays. *Fundam. Math.*, 1978, vol. 99, pp. 205–211.
21. Zhou Youcheng. Covering mapping on solenoids and their dynamical properties. *Chin. Sci. Bull.*, 2000, vol. 45, no. 12, pp. 1066–1070. doi: 10.1007/BF02887175.

22. Charatonik J.J., Covarrubias P.P. On covering mappings on solenoids. *Proc. Am. Math. Soc.*, 2002, vol. 130, no. 7, pp. 2145–2154.

23. Bogatyi S.A., Frolkina O.D. Classification of generalized solenoids. *Trudy seminara po vektornomu i tenzornomu analizu* [Proc. Semin. on Vector and Tensor Analysis]. Vol. XXVI. Moscow, Mosk. Gos. Univ., 2005, pp. 31–59. (In Russian)

24. Gumerov R.N. On finite-sheeted covering mappings onto solenoids. *Proc. Am. Math. Soc.*, 2005, vol. 133, no. 9, pp. 2771–2778. doi: 10.2307/4097643.

25. Brownlowe N., Raeburn I. Two families of Exel-Larsen crossed products. *J. Math. Anal. Appl.*, 2013, vol. 398, no. 1, pp. 68–79. doi: 10.1016/j.jmaa.2012.08.026.

26. Gumerov R.N. Dynamical properties of covering mappings of solenoids. *Proc. Int. Conf. "Function Spaces, Approximation Theory, Nonlinear Analysis" dedicated to the centennial of S.M. Nikolskii (Moscow, Russia, May 23–29, 2005).* Moscow, 2005, p. 92. (In Russian).

27. Eda K., Matijevic V. Covering maps over solenoids which are not covering homomorphisms. *Fundam. Math*, 2013, vol. 221, pp. 69–82. doi: 10.4064/fm221-1-3 .

28. Coburn L.A. The C^*-algebra generated by an isometry. *Bull. Am. Math. Soc.*, 1967, vol. 73, no. 5, pp. 722–726.

29. Douglas R.G. On the C^*-algebra of a one-parameter semigroup ofisometries. *Acta Math.*, 1972, vol. 128, pp. 143–152.

30. Murphy G.J. Ordered groups and Toeplitz algebras. *J. Oper. Theory*, 1987, vol. 18, no. 2, pp. 303–326.

31. Grigoryan S.A., Salakhutdinov A. F. C^*-algebras generated by cancellative semigroups. *Sib. Math. J.*, 2010, vol. 51, no. 1, pp. 12–19. doi: 10.1007/s11202-010-0002-y.

32. Grigoryan T.A., Lipaheva E.V., Tepoyan V.A. On the extension of the Toeplitz algebra. *Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki*, 2012, vol. 154, no. 4, pp. 130–138. (In Russian)

33. Li X. Semigroup C^*-algebras and amenability of semigroups. *J. Funct. Anal.*, 2012, vol. 262, no. 10, pp. 4302–4340. doi: 10.1016/j.jfa.2012.02.020.

34. Lipacheva E.V., Hovsepyan K.H. Automorphisms of some subalgebras of the Toeplitz algebra. *Sib. Math. J.*, 2016, vol. 57, no. 3, pp. 525–531. doi: 10.1134/S0037446616030149.

35. Adji S., Laca M., Nilsen M., Raeburn I. Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups. *Proc. Am. Math. Soc.*, 1994, vol. 122, no. 4, pp. 1133–1141. doi: 10.1090/S0002-9939-1994-1215024-1.

36. Murphy G.J. *C^*-algebras and operator theory.* New York, Academic Press, 1990, 286 p.

37. Gumerov R.N. Limit automorphisms of C^*-algebras generated by isometric representations for semigroups of rational numbers. *Sib. Math. J.*, 2018, vol. 59, no. 1, pp. 73–84. doi: 10.1134/S0037446618010093.

38. Helemskii A.Ya. *Banach and Locally Convex Algebras.* New York, Clarendon Press, Oxford Univ. Press, 1993. 464 p.

Received
October 25, 2017

Gumerov Renat Nelsonovich, Candidate of Physical and Mathematical Sciences, Associate Professor of the Department of Mathematical Analysis

Kazan Federal University
ul. Kremlevskaya, 18, Kazan, 420008 Russia
E-mail: *Renat.Gumerov@kpfu.ru*

УДК 517.986

Накрытия соленоидов и автоморфизмы полугрупповых C^*-алгебр

Р.Н. Гумеров

Казанский (Приволжский) федеральный университет, г. Казань, 420008, Россия

Аннотация

В статье рассматриваются конечнолистные накрывающие отображения P- адических соленоидов и предельные эндоморфизмы полугрупповых C^*-алгебр. Цель нашего изложения двояка. Во-первых, это представление результатов, касающихся таких отображений и эндоморфизмов. Во-вторых, мы демонстрируем доказательства некоторых из них. Показывается, что каждое накрывающее отображение соленоида изоморфно отображению возведения в степень. Мы обсуждаем динамические свойства накрывающих отображений. Отображение возведения в степень для P- адического соленоида является топологически транзитивным. Дается критерий хаотичности накрывающего отображения. Классическая теорема Ферма-Эйлера может быть использована для его доказательства. Далее мы рассматриваем предельные эндоморфизмы C^*-алгебр, порожденных изометрическими представлениями полугрупп рациональных чисел. В теоретико-числовых, алгебраических и функциональных терминах нами формулируются критерии того, что предельные эндоморфизмы будут автоморфизмами. С теоретико-категорной точки зрения дается доказательство необходимости условия в таком критерии.

Ключевые слова: автоморфизм C^*-алгебр, алгебра Теплица, $*$-гомоморфизм, индуктивная последовательность алгебр Теплица, ассоциированная с последовательностью простых чисел, конечнолистное накрывающее отображение, обратные и индуктивные последовательность и предел, полугрупповая C^*-алгебра, тологически транзитивное, хаотическое.

Поступила в редакцию
25.10.17

Гумеров Ренат Нельсонович, кандидат физико-математических наук, доцент кафедры математического анализа

Казанский (Приволжский) федеральный университет
ул. Кремлевская, д. 18, г. Казань, 420008, Россия
E-mail: *Renat.Gumerov@kpfu.ru*

***For citation*:** Gumerov R.N. Coverings of solenoids and automorphisms of semigroup C^*-algebras. *Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki*, 2018, vol. 160, no. 2, pp. 275–286.

***Для цитирования*:** *Gumerov R.N.* Coverings of solenoids and automorphisms of semigroup C^*-algebras // Учен. зап. Казан. ун-та. Сер. Физ.-матем. науки. – 2018. – Т. 160, кн. 2. – С. 275–286.