Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 46, No. 4, pp.22–26, 2002 *UDC 517.518*

AN ORTHONORMED SYSTEM

M.G. Grigoryan

Let us recall the following definitions. Assume that $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in [0,1]$, is an orthonormed system.

Definition 1. A series of the form

$$\sum_{k=1}^{\infty} a_k \varphi_k(x), \quad \sum_{k=1}^{\infty} |a_k| > 0, \tag{1}$$

is called a zero-series in the sense of the convergence almost everywhere (everywhere, by measure, by metric of $L^p_{[0,1]}$, $p > 0$) if it converges to zero almost everywhere (everywhere, by measure, by metric of $L^p_{[0,1]}$, $p > 0$, respectively).

Definition 2. Series (1) is said to be unconditionally convergent if, for any permutation $\{\sigma(k)\}$ of the natural numbers the series $\sum\limits_{k=1}^{\infty} a_{\sigma(k)} \varphi_{\sigma(k)}(x)$ converges.

Definition 3. Series (1) will be called an unconditional zero-series in the sense of the convergence everywhere (in the sense of the convergence by metric of $L^p_{[0,1]}$, $p > 0$, respectively) if it converges unconditionally to zero everywhere on $[0,1]$ (by metric of $L^p_{[0,1]}$, $p > 0$, respectively).

Numerous works (see [1]–[9]) are devoted to the question about the existence of zero-series by a preassigned orthonormed system.

The first trigonometric zero-series in the sense of the convergence almost everywhere was constructed in 1916 by D.Ye. Men'shov (see [1]). In 1956 A.A. Talalyan proved that by any complete orthonormed in $L^2_{[0,1]}$ system $\{\varphi_n(x)\}$ one can always indicate a zero-series in the sense of the convergence by measure.

Further important results in this direction were obtained by P.L. Ul'yanov (see [3]), B.S. Kashin (see [4]), V.A. Skvortsov (see [5]), M.L. Petrovskaya (see [6]), F.G. Arutyunyan (see [7]), G.M. Mushegyan and R.I. Ovsepyan (see [8]).

In this article we prove the following

Theorem 1. *The orthogonal series exists*

$$\sum_{k=1}^{\infty} a_k \omega_k(x), \quad \sum_{k=1}^{\infty} |a_k| > 0,$$

which is an unconditional zero-series in the sense of both the convergence by metric of $L^p_{[0,1]}$ *for all* $p \in (1,2)$, *and the convergence everywhere on* $[0,1]$.

Proof. Let

$$\{f_k(x)\}_{k=1}^{\infty} \tag{2}$$

be a sequence of all algebraical polynomials with rational coefficients. One can easily see that a sequence of functions $\{\beta_k(x)\}_{k=1}^{\infty}$ can be defined such that

$$\beta_k(x) = 0, \quad x \notin \left[\frac{1}{k+1}, \frac{1}{k}\right] = \Delta_k,$$

$$\int_{\Delta_k} |\beta_k|^2 dx < \frac{1}{k^2}; \quad \beta_k(x) \notin \bigcup_{q>2} L^q_{[0,1]}, \quad k \geq 1.$$

We put

$$g_k(x) = f_k(x) + \beta_k(x), \quad x \in [0,1], \quad k \geq 1.$$

Obviously, the system of functions $\{g_k(x)\}_{k=1}^{\infty}$ is linearly independent, closed in $L^2_{[0,1]}$, and, for any numbers N and $\alpha_1, \alpha_2, \dots, \alpha_N$ with $\sum_{k=1}^{N} |\alpha_k| > 0$ we have

$$\sum_{k=1}^{N} \alpha_k g_k(x) \notin \bigcup_{q>2} L^q_{[0,1]}.$$

By passing from this system to its orthogonal and normed form on $[0,1]$, we obtain the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ with the properties

a) the system $\{\varphi_n(x)\}$ is complete in L^2 and orthonormed on $[0,1]$;

b) $\sum_{k=1}^{N} \alpha_k \varphi_k(x) \notin \bigcup_{q>2} L^q_{[0,1]}$ for any numbers N and $\{\alpha_k\}_{k=1}^{N}$ with $\sum_{k=1}^{N} |\alpha_k| > 0$.

Let us show that, for any $N > 1$, the system $\{\varphi_k(x)\}_{k=N+1}^{\infty}$ is closed in all $L^p_{[0,1]}$, $1 \leq p < 2$. Suppose the contrary, i.e., for a certain $p_0 \in [1,2)$ and a certain $N_0 > 1$, the system $\{\varphi_k(x)\}_{k=N_0+1}^{\infty}$ is not closed in $L^{p_0}_{[0,1]}$. Then it *is not complete* with respect to $L^{q_0}_{[0,1]}$ (besides, $\frac{1}{q_0} + \frac{1}{p_0} = 1$ for $p_0 > 1$ and $q_0 = +\infty$ for $p_0 = 1$), i.e., a function exists $g(x) \in L^{q_0}_{[0,1]}$ with $\|g(x)\|_2 > 0$, for which

$$\int_0^1 g(x)\varphi_k(x)dx = 0 \quad \forall k \geq N_0 + 1.$$

Consequently, for all $n \geq 1$, we have

$$\int_0^1 \left[g(x) - \sum_{k=1}^{N_0} \alpha_k \varphi_k(x)\right] \varphi_n(x) dx = 0,$$

where

$$\alpha_k = \int_0^1 g(x)\varphi_k(x)dx, \quad 1 \leq k \leq N_0.$$

In view of the fact that the system $\{\varphi_n(x)\}_{n=1}^{\infty}$ is complete in $L^2_{[0,1]}$ (see property a)), we immediately obtain

$$\sum_{k=1}^{N_0} \alpha_k \varphi_k(x) dx = g(x) \in L^{q_0}_{[0,1]}, \quad \sum_{k=1}^{N_0} \alpha_k^2 = \int_0^1 g^2(x)dx \neq 0.$$

We have arrived at a contradiction (see property b) of the system $\{\varphi_n(x)\}_{n=1}^{\infty}$). Consequently, for any natural $N > 1$, the system $\{\varphi_n(x)\}_{n=N+1}^{\infty}$ is closed in all $L^p_{[0,1]}$, $1 \leq p < 2$.

Using this property of the system $\{\varphi_n(x)\}_{n=1}^{\infty}$, by applying the induction, we choose pairwise non-intersecting polynomials

$$h_k = \sum_{i=N_{k-1}}^{N_k - 1} a_i \varphi_i(x), \quad 1 = N_0 < N_1 < \dots < N_k, \quad k = 1, 2, \dots, \tag{3}$$

so that

$$\|f_k(x) - h_k(x)\|_{p_1} < 2^{-2(k+1)}, \quad k = 1, 2, \dots \tag{4}$$

Suppose that we have already determined the natural numbers $1 = \nu_1 < \nu_2 < \dots < \nu_{s-1}$ and the polynomials $\{h_{\nu_k}(x)\}_{k=1}^{s-1}$, for which we have

$$\|h_{\nu_1}\|_{p_1} > 0, \quad \left\| \sum_{k=1}^{m} h_{\nu_k}(x) \right\|_{p_m} < 2^{-2(m+1)}, \quad 1 \le m \le s-1, \tag{5}$$

where $p_m = 2 - 1/m$. Let us take a function f_{ν_s} in sequence (2) such that

$$\left\| \sum_{k=1}^{s-1} h_{\nu_k}(x) + f_{\nu_s}(x) \right\|_{p_{s+1}} < 2^{-2(s+2)}.$$

Hence and from (4) it follows

$$\left\| \sum_{k=1}^{s} h_{\nu_k}(x) \right\|_{p_{s+1}} < 2^{-2(s+1)}; \tag{6}$$

consequently, we have as well

$$\|h_{\nu_s}(x)\|_{p_s} < 2^{-2s}. \tag{7}$$

Thus, by the induction, we can determine the series

$$\sum_{k=1}^{\infty} c_k \omega_k(x) = \sum_{k=1}^{\infty} h_{\nu_k}(x), \tag{8}$$

where

$$\omega_k(x) = \frac{h_{\nu_k}(x)}{\|h_{\nu_k}\|_2}, \quad c_k = \|h_{\nu_k}\|_2, \tag{9}$$

whose terms are chosen from (3) and satisfy conditions (6) and (7) for all $s \ge 3$.

Let $\{\sigma(s)\}_{s=1}^{\infty}$ be any permutation of natural numbers. Let us prove that the series $\sum_{k=1}^{\infty} c_{\sigma(k)} \omega_{\sigma(k)}(x)$ converges to zero simultaneously in all the metrics of $L^p_{[0,1]}$, $1 \le p < 2$. Indeed, let $p \in [1, 2)$ and $\epsilon > 0$. We take natural numbers k_1, k_0 so large that the following conditions be fulfilled $k_1 > k_0 > \log_2 \frac{1}{\epsilon}$, $p_{k_0} > p$, $\sigma(k) > k_0$, $k > k_1 > k_0$. Hence and from (6), (7), for all $n > k_1$, we have

$$\left\| \sum_{k=1}^{n} c_{\sigma(k)} \omega_{\sigma(k)}(x) \right\|_p \le \left\| \sum_{i=1}^{k_0} c_i \omega_i(x) \right\|_{p_{k_0}} + \sum_{i=k_0+1}^{\infty} \|c_i \omega_i(x)\|_{p_{k_0}} \le 2^{-2k_0} + 2^{-k_0}.$$

Now let us prove that series (8) converges unconditionally to zero everywhere on $[0, 1]$.

To this end we define the measurable sets

$$E_k = \left\{ x \in [0,1] : \left| \sum_{s=1}^{k} h_{\nu_s}(x) \right| < 2^{-(k+1)} \right\}, \quad k \ge 1. \tag{10}$$

Hence and from (5) it follows

$$|h_{\nu_k}(x)| < 2^{-k+1}, \quad x \in E_k \cap E_{k-1}, \tag{11}$$

$$|E_k| > 1 - 2^{-k}. \tag{12}$$

Put

$$E = \bigcup_{s=1}^{\infty} \bigcap_{k=s}^{\infty} E_k. \tag{13}$$

Obviously (see (12)), we have $|E| = 1$. Let $\{\sigma(k)\}_{k=1}^{\infty}$ be an arbitrary permutation of the natural numbers and let $x \in E$; then for a certain natural k_0 we have $x \in E_k$, $k \geq k_0$ (see (13)).

For any given positive $\epsilon > 0$, we take a natural number $n_0(\epsilon)$ so large that

$$n_0 > s_0, \quad \sigma(n) > s_0 \quad \forall n > n_0,$$

where $s_0 = [\log_2 \frac{1}{\epsilon}]$ ($[a]$ stands for the integer part of the number a).

Consequently, for all $n > n_0$, by virtue of (8)–(11), we obtain

$$\left|\sum_{k=1}^{n} c_{\sigma(k)}\omega_{\sigma(k)}(x)\right| \leq \left|\sum_{k=1}^{s_0+1} c_k\omega_k(x)\right| + \sum_{k=s_0+2}^{\infty} |c_k\omega_k(x)| < 2^{-2(s_0+1)} + 2^{-2(s_0+1)} = 2^{-2s_0-1} < \epsilon,$$

i. e., series (8) converges unconditionally almost everywhere on $[0, 1]$ to zero. If now we put

$$\overline{\omega_k}(x) = \begin{cases} \omega_k(x), & x \in [0,1] \setminus E; \\ 0, & x \in E, \end{cases}$$

then we immediately obtain the convergence of the series $\sum c_k\overline{\omega_k}(x)$ to zero on $[0, 1]$. □

Remark. This Theorem is complete in the following sense: First, for none $p \geq 1$ ($p \geq 2$, respectively), by none bounded orthonormed (by none orthonormed, respectively) system $\{\varphi_n(x)\}$ a zero-series in the sense of the convergence by metric of $L^p_{[0,1]}$ exists; second, by the system $\{\psi_k(x)\}$, constructed by B.S. Kashin (see [4]), one cannot indicate zero-series in the sense of the convergence almost everywhere.

From the proof of Theorem 1 it is seen that the orthonormed system (ONS) $\{\omega_k(x)\}_{k=1}^{\infty}$ is an unconditional system of representation of functions of the classes $L^p_{[0,1]}$, $1 \leq p < 2$. More exactly, the following theorem takes place.

Theorem 2. *An orthonormed system* $\{\omega_k(x)\}_{k=1}^{\infty}$ *exists possessing the property*: *For any function* $f(x) \in L^p_{[0,1]}$ *for a fixed* $p \in [1, 2)$ ($f(x) \in \bigcap_{1\leq p<2} L^p_{[0,1]}$, *respectively*), *a series by the system* $\{\omega_k(x)\}$ *of the form* $\sum_{k=1}^{\infty} a_k\omega_k(x)$ *exists which converges unconditionally to* $f(x)$ *both in metric of* $L^p_{[0,1]}$ (*in all metrics of* $L^p_{[0,1]}$, $1 \leq p < 2$, *respectively*), *and almost everywhere on* $[0, 1]$.

In the conclusion, let us formulate the following question.

What are *necessary and sufficient* conditions to be imposed upon ONS $\{\omega_k(x)\}$ in order for one could construct by this system a zero-series (unconditionally zero-series, respectively) in the sense of the convergence by norm of $L^p_{[0,1]}$, $1 \leq p < 2$?

References

1. D Menchoff, Sur l'unicieté du développment trigonometrique, C.R. de l'Acad. des Sci. a Paris, Vol. 163, pp. 433–436, 1916.
2. A.A. Talalyan, On convergence of orthogonal series, DAN SSSR, Vol. 110, pp. 515–516, 1956.
3. P.L. Ul'yanov, Representation of functions by series and the classes $\varphi(L)$, UMN, Vol. 27, Vyp. 2, pp. 3–52, 1972.
4. B.S. Kashin, On one complete orthonormed system, Matem. sborn., Vol. 99, pp. 356–365, 1976.
5. V.A. Skvortsov, One example of zero-series by the Walsh system, Matem. zametki, Vol. 19, pp. 179–186, 1976.
6. M.L. Petrovskaya, On zero-series by the Haar system and uniqueness sets, Izv. AN SSSR. Ser. matem., Vol. 28, no. 4, pp. 773–798, 1964.
7. F.G. Arutyunyan, On the uniqueness of series by the Haar system, DAN Arm. SSR, Vol. 38, no. 3, pp. 129–134, 1964.

8. G.M. Mushegyan and R.I. Ovsepyan, On the uniqueness of orthogonal series, DAN Arm. SSR. Ser. matem., Vol. 4, no. 4, pp. 259–266, 1969.
9. M.Zh. Grigoryan, Representation of functions of classes $L^p[0,1]$, $1 \leq p \leq 2$, by orthogonal series, DAN Arm. SSR, Vol. 67, no. 5, pp. 269–274, 1978.

18 December 2000

Yerevan State University