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ON A BOUND OF A FUNCTIONAL IN THE CLASS OF STARLIKE FUNCTIONS

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We denote by S^* the class of starlike functions $w = f(z)$, $f(0) = 0$, $f'(0) = 1$, which are regular and univalent in the disk $E = \{z \in C : |z| < 1\}$, and map E onto a domain which is starlike with respect to $w = 0$. In [1], the range B_z of a system of functionals $\left\{\ln\left|\frac{f(z)}{z}\right|, \left|\frac{zf'(z)}{f(z)}\right|\right\}$ on the class S^* was studied for a fixed z, $0 < |z| < 1$. In the present article we use these results to improve the upper bound of $\left|\frac{zf'(z)}{f(z)}\right|$ on S^*, dependent on $\ln\left|\frac{f(z)}{z}\right|$, which was obtained in another way by V. Singh in [2] (see theorem 2, inequality (27)).

Since B_z does not depend on $\arg z$, in what follows we shall set $z = r$, $0 < r < 1$, and consider the domain B_r.

We use a well-known (see, e. g., [3], p. 507) integral representation for functions from S^* via the Stieltjes integral $f(z) = z\exp\Big[-2\int\limits_{-\pi}^{\pi}\ln(1 - ze^{it})d\mu(t)\Big]$, where $\mu(t)$ is an increasing real-valued function, $-\pi \le t \le \pi$, $\mu(-\pi) = 0$, $\mu(\pi) = 1$. Set

$$I_1(f) = \ln\left|\frac{f(r)}{r}\right| = -\int_{-\pi}^{\pi}\ln(1 - 2r\cos t + r^2)d\mu(t), \quad I_2(f) = \left|\frac{rf'(r)}{f(r)}\right| = \left|\int_{-\pi}^{\pi}\frac{1+e^{it}r}{1-e^{it}r}d\mu(t)\right|.$$

In order to study the upper bound of the range B_r of $I(f) = I_1(f)+iI_2(f)$ in [1] an auxiliary problem was stated to find in S^* the range B_r^+ of $I^+(f) = I_1(f)+iI_2^+(f)$, where $I_2^+(f) = \int\limits_{-\pi}^{\pi}\left|\frac{1+e^{it}r}{1-e^{it}r}\right|d\mu(t)$, $\mu(t)$ being an increasing real-valued function, $-\pi \le t \le \pi$, $\mu(-\pi) = 0$, $\mu(\pi) = 1$. Clearly, $I_2(f) \le I_2^+(f)$. It can be easily seen that $I_2^+(f) = \int\limits_{-\pi}^{\pi} g^+(r,t)d\mu(t)$, where

$$g^+(r,t) = -\ln(1 - 2r\cos t + r^2) + i\sqrt{\frac{1 + 2r\cos t + r^2}{1 - 2r\cos t + r^2}}. \tag{1}$$

It is well-known (see [4]) that the range of $I^+(f)$ is a convex hull of the curve Γ given by the equation $\xi = g^+(r,t)$, $-\pi \le t \le \pi$. In our case the curve has an inflexion point. Consequently, both the domains B_r^+ and B_r are bounded from above by a part of the curve Γ (this part is denoted by $\Gamma_1 : \xi = g^+(r,t)$, $\alpha \le t \le \pi$, $0 < \alpha < \pi$) and by a tangent segment to Γ, drawn from the point $M\Big(-\ln(1-r)^2, \frac{1+r}{1-r}\Big)$. Thus, the upper part of the boundary of B_r^+ consists of a concave curve Γ_1 and a segment MM_0, where M_0 is a tangency point.