ISSN 1066-369X, Russian Mathematics, 2018, Vol. 62, No. 8, pp. 76–79. © Allerton Press, Inc., 2018.
Original Russian Text © F.G. Avkhadiev, R.G. Nasibullin, I.K. Shafigullin, 2018, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2018, No. 8, pp. 88–92.

# $L_p$-Versions of One Conformally Invariant Inequality

**F. G. Avkhadiev*, R. G. Nasibullin**, and I. K. Shafigullin*****

*Kazan Federal University*
*ul. Kremlyovskaya 18, Kazan, 420008 Russia*

Received April 26, 2018

**Abstract**—We obtain $L_p$-versions of theorems proved by J. L. Fernández and J. M. Rodríguez in the paper "The Exponent of Convergence of Riemann Surfaces, Bass Riemann Surfaces", Ann. Acad. Sci. Fenn. Ser. A. I. Mathematica **15**, 165–182 (1990). An important role in the proof of our results is due to the approach of V. M. Miklyukov and M. K. Vuorinen. In particular, we use the isoperimetric profile of hyperbolic domains.

**DOI:** 10.3103/S1066369X1808011X

Keywords: *Poincaré metric, isoperimetric inequality, uniformly perfect set, Hardy type inequality.*

**Introduction.** Let $\Omega$ be a domain of hyperbolic type in the extended plane $\overline{\mathbb{C}}$. We define the hyperbolic radius by the formula $R(z,\Omega)=1/\lambda_\Omega(z)$ $(z\in\Omega)$, where $\lambda_\Omega$ is the coefficient of the Poincaré metric of Gaussian curvature $k=-4$ ([1, 2]).

Suppose that $p\in[1,\infty)$ and $z=x+iy$. Consider the following variational Hardy type inequality:

$$\iint_\Omega \frac{|\nabla u|^p}{R^{2-p}(z,\Omega)}dx\,dy \ge c_p(\Omega)\iint_\Omega \frac{|u|^p}{R^2(z,\Omega)}dx\,dy \quad \forall u\in C_0^1(\Omega), \tag{1}$$

where $\nabla u$ is the gradient of the function $u$, and

$$c_p(\Omega)=\inf_{u\in C_0^1(\Omega),\ u\not\equiv 0}\iint_\Omega \frac{|\nabla u|^p}{R^{2-p}(z,\partial\Omega)}dx\,dy\left(\iint_\Omega \frac{|u|^p}{R^2(z,\partial\Omega)}dx\,dy\right)^{-1} \tag{2}$$

is a constant defined as the maximum possible constant in the inequality (1).

We suppose that the smoothness of $u(z)$ at the point $z=\infty$ is equivalent to the smoothness of $u(1/z)$ at the point $z=0$.

In the case $p=2$ one has the following conformally invariant inequality:

$$\iint_\Omega |\nabla u|^2 dx\,dy \geqslant c_2(\Omega)\iint_\Omega \frac{|u|^2}{R^2(z,\Omega)}dx\,dy \quad \forall u\in C_0^1(\Omega). \tag{3}$$

This inequality is widely known in the spectral theory of the Laplace–Beltrami operator on the Riemannian manifold of constant negative curvature (see [3] and the bibliography therein). In particular, it is known that $c_2(\Omega)=1$ for every simply or doubly connected domain of hyperbolic type and that $c_2(\Omega)\in[0,1]$ for every domain of hyperbolic type. There are domains for which $c_2(\Omega)=0$, i.e., inequality (3) has no sense. These assertions are consequences of the known facts of the hyperbolic geometry and the following formula of Elstrodt–Patterson–Sullivan ([3], P. 333):

$$c_2(\Omega)=\{1 \text{ for } 0\le\beta\le 1/2,\ \ 4\beta\,(1-\beta)\ \text{ for } 1/2\le\beta\le 1\},$$

where $\beta=\beta(\Omega)$ is the critical exponent of convergence of the Poincaré–Dirichlet series for the fundamental group of transformation of $\Omega$.

---

*E-mail: avkhadiev47@mail.ru.
**E-mail: nasibullinRamil@gmail.com.
***E-mail: shafigullin.ik@gmail.com.