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A Semilattice Generated by Superlow Computably Enumerable Degrees

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Abstract—We prove that a partially ordered set of all computably enumerable (c. e.) degrees that are the least upper bounds of two superlow c. e. degrees is an upper semilattice not elementary equivalent to the semilattice of all c. e. degrees.

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A function g is called *ω-computably enumerable* (c. e.) if there exist computable functions f and h such that for all x,

$$g(x) = \lim_s f(x,s),$$
$$|\{s : f(x,s) \neq f(x,s+1)\}| \leqslant h(x).$$

A set is ω-c. e. if its characteristic function is ω-c. e. A set A is called *superlow* if its Turing jump A' is truth-table reducible to $\emptyset'$ ($A' \leqslant_{tt} \emptyset'$). Due to the result obtained in [1], superlow sets are characterized by the condition that A' is ω-c. e. We call a degree superlow if it contains a superlow set. Superlow degrees are studied in papers papers [2–4]. In [5] the following definition of *totally ω-c. e.* degrees is introduced: A degree $\mathbf{a}$ is called *totallyω-c. e.* if each function $g \leqslant_T \mathbf{a}$ is an ω-c. e. function. It can be easily proved that each superlow c. e. degree is a totally ω-c. e. degree. Notice that due to results described in [6] the inverse inclusion is incorrect. In [5] one establishes the following structural property of computably enumerable totally ω-c. e. degrees: A computably enumerable degree is totally ω-c. e. if and only if it does not bound critical triples. This characterization allowed the authors of [5] to prove the elementary nonequivalence of partially ordered sets of low c. e. degrees and superlow c. e. degrees. In this paper we use a technique for treating totally ω-c. e. degrees for proving (by taking the least upper bound) the elementary nonequivalence of the semilattice of all c. e. degrees and the semilattice generated by superlow c. e. degrees. Denote these semilattices by $\mathbf{C}$ and $\mathbf{J}$, respectively. According to the Sacks splitting theorem [7], every c. e. degree is the least upper bound of two low c. e. degrees. Thus, $\mathbf{C}$ is generated by taking all possible least upper bounds of pairs of low c. e. degrees. In the next theorem we prove that one can generate $\mathbf{J}$ by restricting oneself to the least upper bounds of pairs of superlow degrees. We use the denotations and terminology proposed in [10].

Theorem 1. *Let superlow c. e. degrees $\mathbf{b}_0$, $\mathbf{b}_1$, and $\mathbf{b}_2$ be given. Then there exist superlow c. e. degrees $\mathbf{a}_0$, $\mathbf{a}_1$, and $\mathbf{a}_2$ such that*

$$\mathbf{b}_0 \cup \mathbf{b}_1 \cup \mathbf{b}_2 = \mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{a}_0 \cup \mathbf{a}_2 = \mathbf{a}_1 \cup \mathbf{a}_2.$$

Proof. Without loss of generality we assume that sets B_0, B_1, and B_2 are pairwise disjoint and every of them is infinite. It suffices to build superlow c. e. sets A_0, A_1, and A_2 such that $B_0 \cup B_1 \cup B_2 = A_0 \cup A_1 = A_0 \cup A_2 = A_1 \cup A_2$ and

$$\forall x \in B_0 \cup B_1 \cup B_2 \ (|\{i : x \in A_i\}| = 2).$$

During the proof we use indices i and j as variables with values less than three, and we do k as a variable with values less than two. Let $\{B_{i,s}\}_{s \in \omega}$ be a computable enumeration of sets B_i such that

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