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The Gelfand–Naimark Theorem for C^*-Algebras Over a Ring of Measurable Functions

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1. INTRODUCTION

One of the most important results in the theory of Banach algebras is the Gelfand–Naimark theorem. It describes a commutative C^*-algebra $\mathcal{U}$ over the field of complex numbers $\mathbb{C}$ as an algebra of continuous complex-valued functions defined on the set of pure states of $\mathcal{U}$ endowed with an $*$-weak topology.

The development of the general theory of Banach–Kantorovich C^*-algebras over a ring of measurable functions naturally leads to the question about an analog of the Gelfand–Naimark theorem for such C^*-modules. The structural theory of C^*-modules originates from the work of I. Kaplansky [1], who used these objects in the algebraic approach to the theory of W^*-algebras. Considering C^*-algebras, AW^*-algebras, and W^*-algebras as modules over their centers, one can describe some properties of the mentioned classes of $*$-algebras with the help of Boolean analysis methods (e.g., [2–5]). In particular, in [4] for C^*-algebras, representing modules over the Stone algebra, one obtains vector analogs of Gelfand–Mazur and Gleason–Zhelyazko–Kahan theorems. C^*-modules serve as useful examples of Banach–Kantorovich modules, whose theory is being actively developed now (e.g., [5, 6]). An efficient tool for the study of these Banach–Kantorovich modules, together with the Boolean-valued analysis, is the theory of continuous and measurable Banach bundles [6]. In particular, this enables one to represent a C^*-module over a ring of measurable functions as a measurable bundle of classical C^*-algebras [7]. This, in turn, allows one to establish properties of C^*-modules, "gluing" the corresponding properties of C^*-algebras over the field $\mathbb{C}$. In this paper we apply this approach, proving one version of the Gelfand–Naimark theorem for C^*-modules. We use the terminology and the notation of the theory of Banach–Kantorovich spaces [5] and the theory of measurable bundles stated in [6].

2. PRELIMINARY INFORMATION

Let (Ω, Σ, μ) be a measurable space with a complete finite measure; let $L^0 = L^0(\Omega)$ stand for the algebra of all complex measurable functions defined on (Ω, Σ, μ) (we identify the functions which coincide almost everywhere); let E stand for a complex vector space.

We call a mapping $\|\cdot\| : E \longrightarrow L^0$ an L^0-valued norm on E, if any $x, y \in E$, $\lambda \in \mathbb{C}$ satisfy the correlations

$$\|x\| \geq 0,\ \ \|x\| = 0 \Longleftrightarrow x = 0,\ \ \|\lambda x\| = |\lambda|\,\|x\|,\ \ \|x + y\| \leq \|x\| + \|y\|.$$

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