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ON APPROXIMATION PROPERTIES OF HIGHER DERIVATIVES OF PERIODIC FUNCTIONS

V.E. Geĭt

Monotonely decreasing to zero, every sequence $\{F_n\}_{n=0}^{\infty}$ determines a class F_C of continuous on $(-\infty,+\infty)$ functions f with the period 2π, whose trigonometric approximations satisfy the condition $E_n(f) \le F_n$ $(n = 0, 1, \dots)$. We shall say that the order of the quantity α_n is β_n and write $\alpha_n \sim \beta_n$ if $\alpha_n = O(\beta_n)$ and $\beta_n = O(\alpha_n)$.

Theorem 1. *For an arbitrarily given* r $(= 1, 2, \dots)$ *and any sequence* $F_n \downarrow 0$ $(n \uparrow \infty)$ *we have along the functions* $f \in F_C$ *that*

$$\sup E_n(f^{(r)}) \sim n^r F_n + \sum_{\nu=n+1}^{\infty} \nu^{r-1} F_\nu$$

under assumed convergence of the cited series.

Proof. Let $f \in F_C$ and the series $\sum \nu^{r-1} E_\nu(f) < +\infty$ converge. Then (see, e. g., [1], p. 488) the r-th derivative of $f^{(r)}$ is continuous and

$$E_n(f^{(r)}) = O\Big(n^r E_n(f) + \sum_{\nu=n+1}^{\infty} \nu^{r-1} E_\nu(f)\Big). \tag{1}$$

Hence along the functions f of class F_C we evidently have

$$\sup E_n(f^{(r)}) = O\Big(n^r F_n + \sum_{\nu=n+1}^{\infty} \nu^{r-1} F_\nu\Big). \tag{2}$$

The σ-relation inverse to (2) will be proved as follows. By the given r $(= 1, 2, \dots)$ and sequence $F_n \downarrow 0$, we consider under assumption that $\Delta F_{\nu-1} = F_{\nu-1} - F_\nu$ $(\nu = 1, 2, \dots)$, a function of the form

$$f(x) = f_{r,F}(x) = \sum_{\nu=1}^{\infty} \Delta F_{\nu-1} \cos\Big(\nu x + r\frac{\pi}{2}\Big). \tag{3}$$

Since the deviation of function (3) from its partial sum $S_n(f, x)$ is

$$f(x) - S_n(f, x) = \sum_{\nu=n+1}^{\infty} \Delta F_{\nu-1} \cos\Big(\nu x + r\frac{\pi}{2}\Big),$$

due to the assumption $F_\nu \downarrow 0$ we shall have

$$E_n(f) \le \sum_{\nu=n+1}^{\infty} \Delta F_{\nu-1} = F_n \quad (n = 0, 1, \dots). \tag{4}$$