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APPROXIMATION OF FUNCTIONS BY THE FOURIER–BESSEL SUMS

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1. Let $L_2 = L_2([0,1], x^{2p+1})$ be the space of functions $f : [0,1] \to \mathbb{R}$ integrable with square with the weight x^{2p+1}, $p > -\frac{1}{2}$ and the norm

$$\|f\| = \sqrt{\int_0^1 x^{2p+1} f^2(x)dx}.$$

As is known (see [1], p. 355), the system of functions

$$j_p(\lambda_n x) = \frac{2^p \Gamma(p+1) \Im_p(\lambda_n x)}{(\lambda_n x)^p}, \quad n = 1, 2, \dots,$$

where $\Im_p(u)$ is the Bessel function of the first kind of order p, and $\lambda_1, \lambda_2, \dots$ are enumerated in ascending order positive roots of the equation $\Im_p(u) = 0$, is a complete orthogonal system in the space L_2. We denote by $E_n(f) = \inf\limits_{P_n} \|f - P_n\|$ the best approximation of the function $f \in L_2$ by polynomials of the form

$$P_n(x) = \sum_{i=1}^{n-1} a_i j_p(\lambda_i x).$$

Recall that by the Kolmogorov n-diameter of a set $M \subset L_2$ the following value is called

$$d_n(M) = d_n(M; L_2) = \inf_{F_n \subset L_2} \{ \sup_{f \in M} \{ \inf_{g \in F_n} \|f - g\| \} \},$$

where the last time the greatest lower bound is taken over all subspaces $F_n \subset L_2$ of dimension $n \in \mathbb{N}$ (see [2], p. 186).

In the space L_2 we consider the averaging operator

$$T_h(f) = T_h(f; x) = \frac{\Gamma(p+1)}{\Gamma(\frac{1}{2})\Gamma(p+\frac{1}{2})} \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos t}\,) \sin^{2p} t \, dt, \quad 0 \le h \le 1.$$

Let us note the following properties of this operator (see [3]):

1) $T_h(f)$ is a linear operator,
2) $T_h(j_p(\lambda x)) = j_p(\lambda h) j_p(\lambda x)$,
3) $T_0(f) = T_0(f; x) = f(x)$,
4) $\|T_h(f) - f\| \to 0$, $h \to 0$,
5) the function $u(x,h) = T_h(f;x)$, $x, h \in [0,1]$, is a solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{2p+1}{x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial h^2} + \frac{2p+1}{h}\frac{\partial u}{\partial h},$$
$$u(x,0) = f(x), \quad \frac{\partial u}{\partial h}\bigg|_{h=0} = 0,$$