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ISOPERIMETRIC INEQUALITIES FOR L^p-NORMS OF THE DISTANCE FUNCTION TO THE BOUNDARY

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Abstract

The main goal of the paper is to prove that L^p-norms of $\mathrm{dist}(x, \partial G)$ and $\mathrm{dist}^{-1}(x, \partial G)$ are decreasing functions of p, where G is a domain in $\mathbb{R}^n (n \geq 2)$. We also obtain a sharp estimation of the rate of decreasing for these norms using L^p–norms of the distance function for a consistent ball. We prove a new isoperimetric inequality for L^p–norms of $\mathrm{dist}(x, \partial G)$, this inequality is analogous to the inequality of L^p–norms of the conformal radii (see Avkhadiev F.G., Salahudinov R.G. // J. of Inequal.& Appl. – 2002. – V. 7, No 4. – P. 593–601).

Note that L^2-norm of $\mathrm{dist}(x, \partial G)$ plays an important role to investigate the torsional rigidity in Mathematical Physics (see, for instance, Avkhadiev F.G. // Sbornik: Math. – 1998. – V. 189, No 12. – P. 1739–1748; Bañuelos R., van den Berg M., Carroll T. // J. London Math. Soc. – 2002. – V. 66, No 2. – P. 499–512). As a consequence we get new inequalities in the torsional rigidity problem.

Also we generalize the n-dimensional isoperimetric inequality.

Introduction

Let G be a domain in $\mathbb{R}^n (n \geq 2)$. Let us consider the following geometrical functional [1, 2]

$$\mathrm{I}(p, G) = \int_G \mathrm{dist}^p(x, \partial G)\, dA. \tag{1}$$

Here $\mathrm{dist}(x, \partial G)$ denotes the distance function from $x \in G$ to the boundary ∂G, dA is the volume element $dx_1, \ldots, dx_n$, and $p \geq -1$. In [1] it was noted, that it might be justified to call $\mathrm{I}(p, G)$ the *p-order eulidean moment of G with respect to its boundary.*

First we remark some applications of (1), and further we note connection of (1) with a new conception of *isoperimetrical monotonicity.*

It is clear that 0-order moment is the volume of G. Also, as a limit case of (1), we can get $\mathrm{d}(G) = \max\limits_{x \in G} \mathrm{dist}(x, \partial G)$. We also found that for a wide class of domains (-1)-order moment is, up to a constant factor, the surface area of G.

Further, we remark a new geometrical functional of G that recently appears in the elastic torsion problem. This is the second order eulidean moment or, otherwise, the *eulidean moment of inertia with respect to the boundary.* Consider the boundary value problem

$$\triangle u = -1 \text{ in } G, \quad u = 0 \text{ on } \partial G,$$

where $-\triangle$ is the Dirichlet Laplacian, and let

$$\mathrm{P}(G) := 4 \int_G u(x)\, dA,$$

$\mathrm{P}(G)$ is exactly the torsional rigidity of a simply connected domain G. Just as in [3] we call $\mathrm{P}(G)$ the torsional rigidity of G, even if $n > 2$ and/or G is not simply connected.

We begin from the case $n = 2$. In 1995 F.G. Avkhadiev [1] proved the two-sided inequality

$$\mathrm{I}(2, G) \leq \mathrm{P}(G) \leq 64\, \mathrm{I}(2, G). \tag{2}$$

Later in [4] the left-side hand of the inequality was improved to $3\mathrm{I}(2, G) < 2\, \mathrm{P}(G)$.

Let us remark that an n-dimensional generalization of was recently proved in [5]

$$\frac{2}{n}\mathrm{I}(2, G) \leq \mathrm{P}(G) \leq C_G \mathrm{I}(2, G), \tag{3}$$

under the additional restriction that G satisfies a strong Hardy inequality with some constant (see [6]), where C_G is a functional on G. In particular, the two-sided inequalities (2), (3) answered the question: "*When is the torsional rigidity of a simply connected domain in $\mathbb{R}^n$ bounded?*", however for $n \geq 3$ the question is still open.

Another application of (1) was discovered by F.G. Avkhadiev in [2], this is also generalize (2) on an n-dimentional case. Consider the functional

$$K_{p,q}(G) = \sup_{f \in C_0^\infty(G)} \frac{\left(\iint\limits_G |f|^q(x)\, dA \right)^{1/q}}{\left(\iint\limits_G |\operatorname{grad} f|^p(x)\, dA \right)^{1/p}}.$$

$K_{p,q}(G)$ appears in the Poincare – Sobolev's inequality. In [2] it was proved the two sided estimations for $K_{p,q}(G)$ using (1).

The first property of isoperimetric monotonicity was conjectured by J. Hersch [7], and it was proved by M.-Th. Kohler-Jobin [8]. We shall consider the following boundary value problem [9]

$$\triangle v + \beta v + 1 = 0 \text{ in } G, \quad v = 0 \text{ on } \partial G \quad (-\infty < \beta < \lambda_1(G)) \tag{4}$$

and the corresponding functional

$$Q(\beta) := \int\limits_G v\, dA.$$

In particular, we have $Q(0) = \mathrm{P}(G)/4$, $Q(\lambda_1) = \pi j_0^4 / (8\lambda_1^2(G))$, and $(-\beta)Q(\beta) \underset{\beta \to -\infty}{\longrightarrow} A(G)$ (see, for example, [12]).

By $\overline{Q(\beta)}$ we denote the radius of a ball in $\mathbb{R}^n$ with same $Q(\beta)$.

Theorem A [8, 10]. *Let G be a bounded domain and not a ball in $\mathbb{R}^n$, then $\overline{Q(\beta)}$ is a decreasing function on β. If G is a ball, then $\overline{Q(\beta)}$ is a constant function.*

This monotonicity property contains several well-known plane isoperimetric inequalities (see [12]), which include functionals $\lambda_1(G)$, $\mathrm{P}(G)$, and $A(G)$. On the other hand, existence theorems for $\mathrm{P}(G)$, and $\lambda_1(G)$ are expressing in terms of $\mathrm{I}(\alpha, G)$ (see [1, 11]). Further, other monotonicity properties were discovered by C. Bandle [13], and by J. Hersch [12]. All of these isoperimetric monotonicity properties were connected with the solutions of the differential equations (4). So, the conjecture on the isoperimetric monotonicity property of (1) is a geometrical analog of Theorem A. On the other hand, this work was intended as an attempt to bind continuously the well-known geometrical quantities of a domain in $\mathbb{R}^n$ such as the surface area, the volume, the radius of the largest ball contained in the domain.

1. Main results and corollaries

Let $\alpha > n-1$. Further it would be suitable to use a constant

$$\mathrm{c}_{\alpha,n} := [\mathrm{I}(\alpha - n, B_1)]^{-1} = \frac{\Gamma(n/2)\Gamma(\alpha+1)}{2\pi^{n/2}\Gamma(\alpha-n+1)\Gamma(n)}, \tag{5}$$

where B_1 is a unit ball in $\mathbb{R}^n$, and $\Gamma(\cdot)$ is Euler's Gamma function.

Theorem 1. *Let $\alpha \geq \beta \geq n-1$, and G is a domain in $\mathbb{R}^n$ such that $\mathrm{I}(\alpha-n,G) < < \infty$. Then*

$$\int\limits_G \mathrm{dist}^{\alpha+\beta-n}(x,\partial G)\,dA \leq \frac{\mathrm{c}_{\alpha,n}\mathrm{c}_{\beta,n}}{\mathrm{c}_{\alpha+\beta,n}} \int\limits_G \mathrm{dist}^{\alpha-n}(x,\partial G)\,dA \times \\ \times \int\limits_G \mathrm{dist}^{\beta-n}(x,\partial G)\,dA \tag{6}$$

The equality holds iff G is a ball in $\mathbb{R}^n$.

This kind of inequality was first proved in [14] for conformal moments of a simply connected plane domain. Also, in [14] a chain of plane isoperimetric inequalities was obtained in order to get sharp lower bound in the torsional problem. This chain is similar to '*discrete isoperimetric monotonicity.*' Note that the inequality (6) contains three geometrical characteristics of G.

Corollary 1. *Let $n=2$ and $\alpha=\beta=1$ in Theorem 1, then inequality (6) turns to the plane classical isoperimetric inequality*

$$\mathrm{A}(G) \leq \frac{L^2(G)}{4\pi}.$$

In the next two assertions we prove isoperimetric monotonicity properties for integral functionals which depend on $\mathrm{dist}(x,\partial G)$.

Theorem 2. i) *Let $\|\mathrm{dist}(x,\partial G)\|_p < \infty$, then*

$$\frac{\|\mathrm{dist}(x,\partial G)\|_{p'}}{\|\mathrm{dist}(x,\partial G)\|_{p''}} \leq \frac{\|\mathrm{dist}(x,\partial D_1)\|_{p'}}{\|\mathrm{dist}(x,\partial D_1)\|_{p''}}, \tag{7}$$

where $p' \geq p \geq p'' \geq 0$, $p' > p''$, and D_1 is a ball such that $\|\mathrm{dist}(x,\partial D_1)\|_p := \|\mathrm{dist}(x,\partial G)\|_p$. Equality holds only for a ball in $\mathbb{R}^n$.

ii) *Let $\|\mathrm{dist}^{-1}(x,\partial G)\|_p < \infty$, then*

$$\frac{\|\mathrm{dist}^{-1}(x,\partial G)\|_{p'}}{\|\mathrm{dist}^{-1}(x,\partial G)\|_{p''}} \leq \frac{\|\mathrm{dist}^{-1}(x,\partial D_2)\|_{p'}}{\|\mathrm{dist}^{-1}(x,\partial D_2)\|_{p''}},$$

where $0 \leq p' \leq p \leq p'' < 1$, $p' < p''$, and D_2 is a ball such that $\|\mathrm{dist}^{-1}(x,\partial D_2)\|_p := \|\mathrm{dist}^{-1}(x,\partial G)\|_p$. Equality holds iff G a ball in $\mathbb{R}^n$.

In the plane case for $p = p'' = 0$ the inequality (7) is an analogous of famous St Venant's and Polya's inequality for $\mathrm{P}(G)$. In the case $p = p'' = 0$, and for $p' = 1$ the inequality (7) was proved by J. Leavitt and P. Ungar [15], and in the case $p = p'' = 0$, and $p' \geq 0$ in [16].

Using a result from [4] we can get the following lower estimations for the torsional rigidity of a plane domain.

Corollary 2. *Let G be a simply connected plane domain, and let $\mathrm{P}(G)$ is bounded, then we have for $\alpha \geq 4$*

$$\frac{3\pi}{\alpha(\alpha-1)}\left(\frac{\alpha(\alpha-1)}{2\pi}\int\limits_G \operatorname{dist}^{\alpha-2}(x,\partial G)\,dA\right)^{4/\alpha} < \mathrm{P}(G).$$

Now we shall consider a normalized version of $\mathrm{I}(\alpha-n,G)$

$$\mathbb{I}_\alpha(G) = \left(\mathrm{c}_{\alpha,n}\int\limits_G \operatorname{dist}^{\alpha-n}(x,\partial G)\,dA\right)^{1/\alpha}, \tag{8}$$

where $\alpha > n-1$. In the Lebesgue's sense $\mathbb{I}_{n-1}(G)$ does not exist, however we give a meaningful definition for the case $\alpha = n-1$. In that case we suppose that G has a smooth boundary ∂G, which belongs to $C^1(G)$, and we put

$$\mathbb{I}_{n-1}(G) := \lim_{\alpha\to(n-1)+0}\mathbb{I}_\alpha(G) = \left[\frac{\Gamma(n/2)}{2\pi^{n/2}}\int\limits_{\partial G} dS\right]^{1/(n-1)}, \tag{9}$$

where dS is the volume element of ∂G. On the other hand, we have

$$\lim_{\alpha\to\infty}\mathbb{I}_\alpha(G) = \mathrm{d}(G). \tag{10}$$

Therefore, $\mathrm{d}(G) < \infty$ is the necessary condition to apply our theorems. Further we prove that the evaluations given in (9) and (10) are well defined.

Theorem 3. *Let G is a domain in $\mathbb{R}^n$, and suppose that $\mathbb{I}_{\alpha_0}(G)$ is bounded for some $\alpha_0 \geq n-1$. Then*

i) *If G is not a ball in $\mathbb{R}^n$, then $\mathbb{I}_\alpha(G)$ is a strictly decreasing function of α for $\alpha \geq \alpha_0$ and $\mathbb{I}_\infty(G) = \mathrm{d}(G)$.*

ii) *If G is a ball in $\mathbb{R}^n$, then $\mathbb{I}_\alpha(G)$ is the radius of the ball G for all $\alpha \geq n-1$.*

Corollary 3. *Let G is a domain in $\mathbb{R}^n$ such that $\mathbb{I}_\alpha(G) < \infty$ for $\alpha \in [n-1,n)$, then*

$$\mathbb{I}_n(G) \leq \mathbb{I}_\alpha(G), \ \textit{and}\ \mathbb{I}_\alpha(G) \leq \mathbb{I}_{n-1}(G).$$

In the both equalities the equalities hold only for a ball in $\mathbb{R}^n$.

The last inequalities are a generalization of the classical n-dimentional isoperimetric inequality, which was proved by E. Schmidt [17], moreover, our assertion is a more sharper result.

2. Proofs of theorems

We begin by proving some simple important properties of the functional $\mathrm{I}(\alpha,G)$ for $\alpha > -1$.

Throughout the proof we will use the following notations

$$\begin{aligned} G_\lambda(\alpha) &:= \{x \in G\,|\operatorname{dist}^\alpha(x,\partial G) > \lambda\}, \\ \Gamma_\lambda(\alpha) &:= \{x \in G\,|\operatorname{dist}^\alpha(x,\partial G) = \lambda\} \\ \mathrm{a}(G) &:= \int\limits_G dA, \end{aligned} \tag{11}$$

and denote by $[\cdot]_S$ the Schwarz symmetrization of a domain in $\mathbb{R}^n$, and of a function over the domain. We will usually fix a parameter α in our proof, and in those cases, for brevity, we will drop α, for example, $G_\lambda := G_\lambda(\alpha)$.

Lemma 1. *Let $\alpha > 0$, and G be an unbounded domain in $\mathbb{R}^n$ such that* $\mathrm{I}(\alpha, G) < < \infty$. *Then:* 1) $\mathrm{a}(G_\lambda) < \infty$ *for* $0 < \lambda \le \mathrm{d}^\alpha(G)$, *in particular,* $\mathrm{d}(G) < \infty$*;* 2) $\lambda\, \mathrm{a}(G_\lambda) \underset{\lambda\to 0}{\longrightarrow} 0$.

Proof. Let $\lambda > 0$, then we have a chain of inequalities

$$\mathrm{I}(\alpha, G) \ge \int\limits_{G_\lambda} \mathrm{dist}^\alpha(x, \partial G)\, dA \ge \mathrm{a}(G_\lambda) \inf_{x\in G_\lambda} \mathrm{dist}^\alpha(x, \partial G) = \lambda \mathrm{a}(G_\lambda),$$

so $\mathrm{a}(G_\lambda) \le \lambda^{-1}\, \mathrm{I}(\alpha, G) < \infty$.

Further, note that

$$\mathrm{I}(\alpha, G) = \int\limits_{G_\lambda} \mathrm{dist}^\alpha(x, \partial G)\, dA + \int\limits_{G\setminus G_\lambda} \mathrm{dist}^\alpha(x, \partial G)\, dA.$$

Using the definition of integral by Lebesgues, and integration by parts, we obtain

$$\int\limits_{G_\lambda} \mathrm{dist}^\alpha(x, \partial G)\, dA = \int\limits_0^{\mathrm{a}(G_\lambda)} \lambda(a)\, da = \lambda\, \mathrm{a}(G_\lambda) + \int\limits_\lambda^{\mathrm{d}^\alpha(G)} \mathrm{a}(G_t)\, dt.$$

The last integral bounded by $\mathrm{I}(\alpha, G)$. Hence $\mathrm{a}(G_\lambda)$ is integrable on $[0, \mathrm{d}^\alpha(G)]$ because $\mathrm{a}(G_\lambda) \ge 0$ for all admissible λ. Therefore

$$0 \le \lim_{\lambda\to 0} \lambda\, \mathrm{a}(G_\lambda) \le \lim_{\lambda\to 0} \int\limits_0^\lambda \mathrm{a}(G_t)\, dt = 0.$$

$\square$

In the sequel we will use some well-known properties of the level sets (see, for example, [7]), in particular, we will frequently make use of the relations

$$\begin{aligned}
&\int\limits_G \mathrm{dist}^\beta(x, \partial G)\, dA = \int\limits_0^{\mathrm{d}^\beta(G)} \mathrm{a}(G_\lambda(\beta)) d\lambda \quad \text{for} \quad \beta > 0,\\
&\int\limits_G \mathrm{dist}^\beta(x, \partial G)\, dA = \mathrm{a}(G)\mathrm{d}^\beta(G) + \int\limits_{\mathrm{d}^\beta(G)}^\infty \mathrm{a}(G_\lambda(\beta))\, d\lambda \quad \text{for} \quad \beta < 0,\\
&\frac{d\mathrm{a}(G_\lambda(1))}{d\lambda} = -\int\limits_{\Gamma_\lambda(1)} dS.
\end{aligned} \tag{12}$$

Lemma 2. *Let G be not a ball in $\mathbb{R}^n$, and $[\cdot]_S$ is the Schwarz symmetrization of a domain G_λ. Then:*

1) $\displaystyle\int\limits_{G_\lambda} \mathrm{dist}^\alpha(x, \partial G)\, dA < \int\limits_{[G_\lambda]_S} \mathrm{dist}^\alpha(x, \partial [G]_S)\, dA$ *for* $\alpha > 0$;

2) $\displaystyle\int\limits_{G_\lambda} \mathrm{dist}^\alpha(x, \partial G)\, dA > \int\limits_{[G_\lambda]_S} \mathrm{dist}^\alpha(x, \partial [G]_S)\, dA$ *for* $-1 < \alpha < 0$.

Proof. Let $x \in G_\lambda$, then it is easy to check that

$$\operatorname{dist}(x, \partial G) = \operatorname{dist}(x, \partial G_\lambda) + \lambda^{1/\alpha}. \tag{13}$$

The assertion follows by means of the Schwarz symmetrization of G_λ. Indeed, we have $[\operatorname{dist}(\cdot, \partial G_\lambda)]_S(x) < \operatorname{dist}(x, \partial [G_\lambda]_S)$ for $x \in [G_\lambda]_S$, and $\mathrm{a}([G_\lambda]_S) \leq \mathrm{a}(([G]_S)_\lambda)$ (see [7]). Using (13) we obtain

$$[\operatorname{dist}(\cdot, \partial G)]_S(x) = \left[\operatorname{dist}(\cdot, \partial G_\lambda) + \lambda^{1/\alpha}\right]_S(x) = [\operatorname{dist}(\cdot, \partial G_\lambda)]_S(x) + \lambda^{1/\alpha} <$$
$$< \operatorname{dist}(x, \partial [G_\lambda]_S) + \lambda^{1/\alpha} \leq \operatorname{dist}(x, \partial([G]_S)_\lambda) + \lambda^{1/\alpha} = \operatorname{dist}(x, \partial [G]_S).$$

Using the basic properties of the Schwarz symmetrization, for $\alpha > 0$, we get

$$\int\limits_{G_\lambda} \operatorname{dist}^\alpha(x, \partial G)\, dA = \int\limits_{[G_\lambda]_S} \left([\operatorname{dist}(\cdot, \partial G)]_S(x)\right)^\alpha\, dA < \int\limits_{[G_\lambda]_S} \operatorname{dist}^\alpha(x, \partial [G]_S)\, dA.$$

In the case $-1 < \alpha < 0$ we get the same inequality, but with the revers sign.

□

We now define a ball $D(\subset \mathbb{R}^n)$, which corresponds to the domain G. From (5) we can conclude that $\mathrm{I}(\alpha, D)$, for fixed α, is a strictly increasing function of the radius of D, and runs from 0 to ∞ with the radius of D. Therefore there is exactly one ball D, up to an Euclidean motion, such that $\mathrm{I}(\alpha, D) = \mathrm{I}(\alpha, G)$. Note that D depends of α and G, and it plays very important role in our proof.

According to Lemma 2 we shall distinguish two cases: 1) $\alpha \geq 0$, and 2) $-1 < \alpha < 0$.

Proposition 1. *Let $\alpha \geq 0$, and D is the ball defined as above, then*

$$1)\ \mathrm{I}(\gamma, G) \leq \mathrm{I}(\gamma, D) \quad \textit{for } \gamma > \alpha; \tag{14}$$

$$2)\ \mathrm{I}(\gamma, G) \geq \mathrm{I}(\gamma, D) \quad \textit{for } 0 \leq \gamma < \alpha. \tag{15}$$

In the both cases the equalities hold if and only if G is a ball.

Proof. To prove the assertion we apply the M.-Th. Kohler-Jobin symmetrization with a slight modification. This method was introduced in [8, 10], and it was applied to study isoperimetric properties of the solutions of (4), $\lambda_1(G)$, and the first eigenfunction of (4) (see [3, 8, 10, 12]).

First of all we note that the case $\alpha = 0$ is the special case of Lemma 2 with $\lambda = 0$. Therefore let us fix $\alpha > 0$. Further note that the case $\gamma = 0$ also is another interpretation of Lemma 2 with $\lambda = 0$. Indeed, we have

$$\mathrm{I}(\alpha, D) = \mathrm{I}(\alpha, G) \leq \mathrm{I}(\alpha, [G]_S),$$

and using (5), we obtain $\mathrm{I}(0, D) = \mathrm{a}(D) \leq \mathrm{a}([G]_S) = \mathrm{a}(G) = \mathrm{I}(0, G)$.

Thus we can suppose $\gamma > 0$.

Set

$$i(\lambda) = \int\limits_{G_\lambda} \operatorname{dist}^\alpha(x, \partial G)\, dA - \lambda\, \mathrm{a}(G_\lambda).$$

The corresponding value for D we denote by $i^*(\lambda^*)$.

From (12) we obtain

$$i(\lambda) = \int\limits_{\lambda}^{\mathrm{d}^\alpha(G)} \mathrm{a}(G_t)\, dt,$$

and since $i(\mathrm{d}^\alpha(G)) = 0$, we have

$$-\frac{di(\lambda)}{d\lambda} = \mathrm{a}(G_\lambda). \tag{16}$$

An analogous computation for D_{λ^*} leads to

$$-\frac{di^*(\lambda^*)}{d\lambda^*} = \mathrm{a}(D_{\lambda^*}).$$

Now let us define a correspondence between G_λ and D_{λ^*} by requiring that $i(\lambda) = = i^*(\lambda^*)$. By the definition of D, for $\lambda = \lambda^* = 0$ we have $i(0) = \mathrm{I}(\alpha, G) = \mathrm{I}(\alpha, D) = = i^*(0)$.

Because G_λ is bounded, by Lemma 1, applying Lemma 2 to G_λ and D_{λ^*}, we obtain $\mathrm{a}(D_{\lambda^*}) \le \mathrm{a}(G_\lambda)$. Let $\lambda(i)$ be the inverse of $i(\lambda)$. Hence by the defined correspondence we must have the inequality

$$-\frac{d\lambda^*(i)}{di} \ge -\frac{d\lambda(i)}{di}.$$

Again, using $\lambda^*(\mathrm{I}(\alpha, D)) = 0 = \lambda(\mathrm{I}(\alpha, G))$, we obtain by integration from $i_0 (> 0)$ to $\mathrm{I}(\alpha, G)$

$$\lambda^*(i) \ge \lambda(i). \tag{17}$$

In particular, we have the inequality $\mathrm{d}(D) \ge \mathrm{d}(G)$, which, indeed, follows immediately form Lemma 2 and the definition of D.

From the definition (11) easily follows $G_\lambda(\beta) = G_{\lambda^{\alpha/\beta}}$, where $\beta > 0$. Using Lemma 1, (11), (12), and the equality $-di(\lambda) = \mathrm{a}(G_\lambda)d\lambda$, by (16), we get

$$\int\limits_G \mathrm{dist}^\gamma(x, \partial G)\mathrm{d}A = \int\limits_0^{\mathrm{a}(G)} \lambda(\mathrm{a})d\mathrm{a} = \int\limits_0^{\mathrm{d}^\gamma(G)} \mathrm{a}(G_t(\gamma))\, dt = \int\limits_0^{\mathrm{d}^\gamma(G)} \mathrm{a}(G_{t^{\alpha/\gamma}})\, dt =$$
$$= \frac{\gamma}{\alpha} \int\limits_0^{\mathrm{d}^\alpha(G)} t^{\gamma/\alpha - 1} \mathrm{a}(G_t)\, dt = -\frac{\gamma}{\alpha} \int\limits_0^{\mathrm{d}^\alpha(G)} t^{\gamma/\alpha-1}\, di(t) = \frac{\gamma}{\alpha} \int\limits_0^{\mathrm{I}(\alpha,G)} \lambda^{\gamma/\alpha-1}(i)\, di. \tag{18}$$

In the both cases $\gamma \ge \alpha$, and $0 < \gamma < \alpha$, using (17), we obtain (14), and (15), but in the second case we suppose that $\mathrm{I}(\gamma, G) < \infty$. The cases of equality immediately follows from Lemma 2. $\square$

Proposition 2. *Let $-1 < \alpha < 0$, $\gamma \le 0$, and D is the ball defined in the same way like above, then*

$$1)\ \mathrm{I}(\gamma, G) \le \mathrm{I}(\gamma, D) \quad \textit{for} \quad 0 \ge \gamma > \alpha;$$

$$2)\ \mathrm{I}(\gamma, G) \ge \mathrm{I}(\gamma, D) \quad \textit{for} \quad \alpha > \gamma.$$

Proof. We will use same ideas as in Proposition 1, but in quite different situation.

First, we note that if we fix α, then $\mathrm{I}(\alpha, D)$ is an increasing function of the radius of D. Moreover, the direct calculation shows that the same assertion remains true for $\mathrm{a}(D_\lambda)$ with fixed α and λ. We will use this remark bellow. In particular, from Lemma 2, and the definition of D, follow

$$\mathrm{a}(G) = \mathrm{a}([G]_S) \le \mathrm{a}(D). \tag{19}$$

As in Proposition 1 the case $\gamma = 0$ is the particular case of Lemma 2 with $\lambda = 0$. Therefore let $\gamma < 0$.

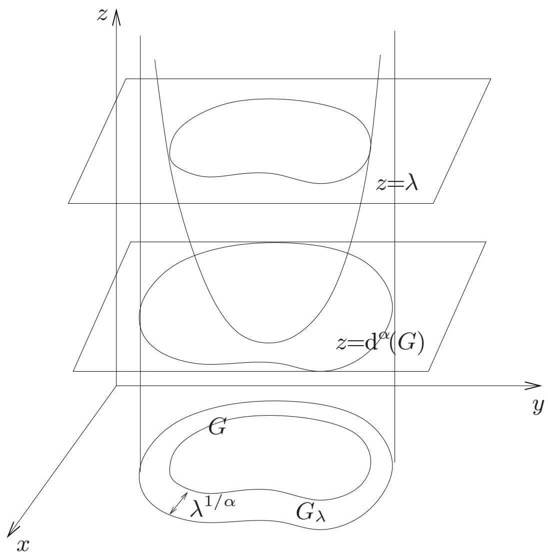

Fig. 1

Set

$$i(\lambda) = \int_{\lambda}^{\infty} \widehat{\mathrm{a}}(G_t)\, dt,$$

where

$$\widehat{\mathrm{a}}(G_\lambda) = \begin{cases} \mathrm{a}(G) & \text{for } \; 0 \le \lambda \le \mathrm{d}^{\alpha}(G), \\ \mathrm{a}(G_\lambda) & \text{for } \; \lambda > \mathrm{d}^{\alpha}(G). \end{cases}$$

In particular, using (12) we get

$$i(0) = \mathrm{a}(G)\mathrm{d}^{\alpha}(G) + \int_{\mathrm{d}^{\alpha}(G)}^{\infty} \mathrm{a}(G_t)\, dt = \mathrm{I}(\alpha, G).$$

For the convenience of the reader we give an illustration for the plane case (see Fig. 1).

The corresponding value for D we denote by $i^*(\lambda^*)$.

Further, almost everywhere we have

$$-\frac{di(\lambda)}{d\lambda} = \widehat{\mathrm{a}}(G_\lambda). \tag{20}$$

An analogous computation for D_{λ^*} leads to

$$-\frac{di^*(\lambda^*)}{d\lambda^*} = \widehat{\mathrm{a}}(D_{\lambda^*}).$$

Let us define a correspondence between G_λ and D_{λ^*} by requiring that $i(\lambda) = i^*(\lambda^*)$. By the definition of D, for $\lambda = \lambda^* = 0$ we have $i(0) = \mathrm{I}(\alpha, G) = \mathrm{I}(\alpha, D) = i^*(0)$.

Applying Lemma 2 to G_λ and D_{λ^*}, the note at the beginning of the proof, and (19), we obtain $\widehat{\mathrm{a}}(D_{\lambda^*}) \geq \widehat{\mathrm{a}}(G_\lambda)$. Let $\lambda(i)$ be the inverse of $i(\lambda)$. Hence by the defined correspondence we must have the inequality

$$-\frac{d\lambda^*(i)}{di} \leq -\frac{d\lambda(i)}{di}.$$

Using $\lambda^*(\mathrm{I}(\alpha, D)) = 0 = \lambda(\mathrm{I}(\alpha, G))$, we obtain by integration from $i_0(>0)$ to $\mathrm{I}(\alpha, G)$

$$\lambda^*(i) \leq \lambda(i). \tag{21}$$

From the definition (11) easily follows $G_\lambda(\beta) = G_{\lambda^{\alpha/\beta}}$, where $\beta < 0$. Using Lemma 1, (11), (12), and the equality $-di(\lambda) = \widehat{\mathrm{a}}(G_\lambda)d\lambda$, by (20), we get

$$\int\limits_G \operatorname{dist}^\gamma(x, \partial G)\, dA = \mathrm{a}(G)\mathrm{d}^\gamma(G) + \int\limits_{\mathrm{d}^\gamma(G)}^\infty \mathrm{a}(G_t(\gamma))\, dt = \mathrm{a}(G)\left(\mathrm{d}^\alpha(G)\right)^{\gamma/\alpha} +$$

$$+ \int\limits_{\mathrm{d}^\gamma(G)}^\infty \mathrm{a}(G_{t^{\alpha/\gamma}})\, dt = \frac{\gamma}{\alpha} \int\limits_0^{\mathrm{d}^\alpha(G)} t^{\gamma/\alpha-1}\mathrm{a}(G)\, dt + \frac{\gamma}{\alpha} \int\limits_{\mathrm{d}^\alpha(G)}^\infty t^{\gamma/\alpha-1}\mathrm{a}(G_t)\, dt =$$

$$= \frac{\gamma}{\alpha} \int\limits_0^\infty t^{\gamma/\alpha-1}\widehat{\mathrm{a}}(G_t)\, dt = -\frac{\gamma}{\alpha} \int\limits_0^\infty t^{\gamma/\alpha-1}\, di(t) = \frac{\gamma}{\alpha} \int\limits_0^{\mathrm{I}(\alpha,G)} (\lambda(i))^{\gamma/\alpha-1}\, di. \tag{22}$$

Let $\alpha < \gamma < 0$, then $\gamma/\alpha - 1 < 0$. From (21) and (22) we easy obtain

$$\mathrm{I}(\gamma, G) = \frac{\gamma}{\alpha} \int\limits_0^{\mathrm{I}(\alpha,G)} (\lambda(i))^{\gamma/\alpha-1} di \leq \frac{\gamma}{\alpha} \int\limits_0^{\mathrm{I}(\alpha,D)} (\lambda^*(i))^{\gamma/\alpha-1}\, di = \mathrm{I}(\gamma, D),$$

which is the first inequality of our proposition. If we interchange α and γ, then the second inequality follows from the first.

This completes the proof of Proposition 2. □

Proof of Theorem 1. We just apply Proposition 1 and Proposition 2 for the case $\mathrm{I}(\alpha, D) = \mathrm{I}(\alpha, G)$. □

Proof of Theorem 2. To prove the first part we apply Proposition 1 for the case $\mathrm{I}(p, D) = \mathrm{I}(p, D_1)$. The second part follows from Proposition 2 for the case $\mathrm{I}(p, D) = = \mathrm{I}(p, D_2)$. □

Proof of Theorem 3. First note that the isoperimetric inequality $\mathbb{I}_\beta(G) \geq \mathbb{I}_\delta(G)$ for $\beta \leq n \leq \delta$ follows easily from Lemma 2, and moreover, we have the same lower and the upper bound $\mathbb{I}_n(G)$ for $\mathbb{I}_\beta(G)$ and $\mathbb{I}_\delta(G)$ respectively. Now we prove that the definition (9) is well-defined for domains with smooth boundaries. For the brevity we will use the denotation $\lim\limits_\alpha := \lim\limits_{\alpha \to (n-1)+0}$. Indeed, using (5), (11), and (12), and applying a integration by parts, we obtain

$$\lim_{\alpha}(\mathbb{I}_\alpha(G))^\alpha = \lim_{\alpha} \mathrm{c}_{\alpha,n} \int\limits_G \mathrm{dist}^{\alpha-n}(x,\partial G)\, dA =$$

$$= \lim_{\alpha} \mathrm{c}_{\alpha,n} \int\limits_0^{\mathrm{a}(G)} \lambda\, (\mathrm{a}(G_\lambda(1)))^{\alpha-n}\, d\mathrm{a}(G_\lambda(1)) = \lim_{\alpha} \frac{\mathrm{c}_{\alpha,n}}{\alpha-n+1} \int\limits_0^{\mathrm{d}(G)} \int\limits_{\Gamma_\lambda(1)} dS d\lambda^{\alpha-n+1} =$$

$$= \lim_{\alpha} \frac{\Gamma(\alpha+1)\Gamma(n/2)}{2\pi^{n/2}\Gamma(\alpha-n+2)\Gamma(n)} \times$$

$$\times \left[\mathrm{d}(G)^{\alpha-n+1} \int\limits_{\Gamma_{\mathrm{d}(G)}(1)} dS \int\limits_0^{\mathrm{d}(G)} \lambda^{\alpha-n+1} d\left(\int\limits_{\Gamma_\lambda(1)} dS \right) \right] = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int\limits_{\partial G} dS.$$

Note that, from (5) and (8) we can see $\mathbb{I}_\alpha(D) = R$ for all $\alpha \ge n-1$, where R is the radius of the ball D.

Let G is not a ball in $\mathbb{R}^n$, and fix $\alpha(\ge \alpha_0)$, then $\mathbb{I}_\alpha(G)$ is bounded, by Proposition 1 and Proposition 2. Like above we can show that there is exactly one ball $D(\alpha)$ in $\mathbb{R}^n$, up to an Euclidean motion, such that $\mathbb{I}_\alpha(G) = \mathbb{I}_\alpha(D(\alpha))$. Further we conclude from (5) and (6) that $\mathbb{I}_\gamma(D(\alpha)) = \mathbb{I}_\delta(D(\alpha))$ for all admissible γ and δ. Therefore for a small $\varepsilon > 0$, from Proposition 1 and Proposition 2 follow again that $\mathbb{I}_{\alpha+\varepsilon}(G)$ is bounded, and the ball $D(\alpha)$ gives a larger $\mathbb{I}_{\alpha+\varepsilon}(\cdot)$ than the domain G, that is

$$\mathbb{I}_{\alpha+\varepsilon}(G) < \mathbb{I}_{\alpha+\varepsilon}(D(\alpha)) = \mathbb{I}_\alpha(D(\alpha)) = \mathbb{I}_\alpha(G), \tag{23}$$

which is the desired conclusion.

To complete the proof we have to prove (10). From the monotonic property of $\mathbb{I}_\alpha(G)$ with respect to the domain we get the inequality $\mathbb{I}_\alpha(G) \ge \mathrm{d}(G)$. Thus it would be enough for us to establish the reverse inequality. Let $\mathbb{I}_{\alpha_0}(G) < \infty$ for some $\alpha_0 \ge n$. Then, applying (18) for $\alpha > \alpha_0$, we can write the equality

$$\mathbb{I}_\alpha(G) = (\mathrm{c}_{\alpha,n}\, \mathrm{I}(\alpha-n))^{1/\alpha} = \left(\mathrm{c}_{\alpha,n} \frac{\alpha-n}{\alpha_0-n} \int\limits_0^{\mathrm{I}(\alpha_0-n,G)} \lambda^{(\alpha-\alpha_0)/(\alpha_0-n)}(i)\, di \right)^{1/\alpha} =$$

$$= \mathrm{d}(G) \left(\frac{(\alpha-n)\mathrm{c}_{\alpha,n}}{(\alpha_0-n)\,[\mathrm{d}(G)]^{-\alpha_0}} \int\limits_0^{\mathrm{I}(\alpha_0-n,G)} \left[\frac{\lambda}{\mathrm{d}^{\alpha_0-n}(G)} \right]^{(\alpha-\alpha_0)/(\alpha_0-n)} (i)\, di \right)^{1/\alpha},$$

where λ corresponds to the level sets of $\mathrm{dist}^{\alpha_0-n}(x,\partial G)$. Thus we easily get

$$\mathbb{I}_\alpha(G) \le \mathrm{d}(G) \left(\frac{(\alpha-n)\ \mathrm{c}_{\alpha,n}}{(\alpha_0-n)\,[\mathrm{d}(G)]^{-\alpha_0}}\ \mathrm{I}(\alpha_0-n,G) \right)^{1/\alpha}.$$

Tending α to the infinity, we get $\mathbb{I}_\alpha(G) \le \mathrm{d}(G)$.

This finishes the proof of the theorem. □

In the conclusion we have to make a remark about reverse inequalities in the theorems. Note that, using Proposition 1 we also can 'go back' from $\mathbb{I}_\alpha(G)$ to $\mathbb{I}_{\alpha-\varepsilon}(G)$, but we need to know that $\mathbb{I}_{\alpha-\varepsilon}(G)$ is bounded. We now give a simple example of domain G on the plane, such that it has $\mathbb{I}_\alpha(G)=\infty$ and $\mathbb{I}_\gamma(G)$ is bounded for $\gamma>\alpha$. In particular, that means, we cannot prove the reverse inequality $\mathbb{I}_\alpha(G)>c\mathbb{I}_\delta(G)$ for $\alpha>\delta$ without additional restrictions on G, here $c>0$ is a universal constant.

Without loss of generality we suppose $\alpha>0$, and consider the domain placed between curves $x=1$, $y=0$, and $y=x^{-1/\alpha}$. There exists $X\in\mathbb{R}$ such that $\operatorname{dist}(z,\partial G)=y/2$, where $z=x+iy$, and $x>X$. Denote by G_X the subdomain of G between $x=1$ and $x=X$, we obtain

$$\int\limits_G \operatorname{dist}^\alpha(x,\partial G)\,dA \approx \int\limits_{G_X} \operatorname{dist}^\alpha(x,\partial G)\,dA + \frac{1}{2}\int\limits_X^{X_0}\frac{dx}{x} \underset{X_0\to\infty}{\longrightarrow} \infty,$$

nevertheless

$$\int\limits_G \operatorname{dist}^{\alpha+\varepsilon}(x,\partial G)\,dA \le \int\limits_{G_X} \operatorname{dist}^{\alpha+\varepsilon}(x,\partial G)\,dA + \int\limits_X^{\infty}\frac{dx}{x^{1+\varepsilon/\alpha}} < \infty,$$

where $\varepsilon>0$. The generalization to an n-dimensional case is obvious.

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Резюме

Р.Г. Салахудинов. Изопериметрические неравенства для L^p-норм функции расстояния до границы области.

Доказана изопериметрическая монотонность евклидовых степенных моментов области относительно своей границы. Доказанное свойство, эквивалентно изопериметрическим неравенствам для L^p-норм функции расстояния до границы области для различных значений p.

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