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Symmetric Solutions of Inverse Boundary-Value Problems and Their Univalence Conditions

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In inverse boundary-value problems (IBVPs) one has to construct a domain D_z and a function $w(z)$, $z = x + iy$, using the known boundary values. This leads to the following question: In which way conditions imposed on the boundary values affect D_z and $w(z)$? For example, one can seek for a domain D_z which satisfies the symmetry conditions (bilateral, rotational, or with respect to the unit circumference). One of the first works dedicated to this question was paper [1]. In the mentioned paper L. A. Aksent'ev constructs a solution to the inner and outer IBVPs with respect to a parameter s, possessing rotational and bilateral symmetric properties. In this paper we consider the necessary and sufficient conditions for various symmetry types for IBVPs with respect to parameters x, $\theta = \arg z$, $r = |z|$, (x, y), and (x, y, θ). For outer IBVPs we obtain certain sufficient univalence conditions, using the approach described in [2].

Let a domain D in a plane z and the image $\Delta = f(D)$ possess the bilateral symmetry with respect to the straight line inclined at the angle $\alpha \in [-\pi/2, \pi/2]$ towards the positive direction of the real axis. This symmetry admits the following analytic representation:

$$f(e^{2i\alpha}\overline{z}) = e^{2i\alpha}\overline{f(z)}. \tag{1}$$

Taking into account (1), consider a Hölder function $u(\gamma)$ defined on the segment $[\alpha - \pi, \alpha + \pi]$ subject to $u(\alpha - \pi) = u(\alpha + \pi)$, the Schwarz integral in the unit circle E

$$f(z) = \frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} u(\gamma)\frac{e^{i\gamma} + z}{e^{i\gamma} - z}d\gamma, \tag{2}$$

and the function

$$v(\gamma) = \operatorname{Im} f(e^{i\gamma}) = -\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} u(s)\cot\frac{s-\gamma}{2}ds.$$

Lemma 1. *The function $f(z)$ defined by formula* (2) *satisfies the symmetry condition* (1)*, if and only if the following correlation is valid*:

$$u(\alpha - \gamma) = u(\alpha + \gamma)\cos 2\alpha + v(\alpha + \gamma)\sin 2\alpha.$$

We say that the function $f(z)$ satisfies the n-symmetry condition in a domain D, if

$$f(e^{2\pi i/n}z) = e^{2\pi i/n}f(z) \ \ \forall z \in D. \tag{3}$$

Lemma 2. *If the function $f(z)$ satisfies condition* (1) *with $\alpha = 0$ and $\alpha = \pi/n$, then it satisfies condition* (3).

Let $s(z)$ stand for one of the following symmetries: $s(z) = e^{2i\alpha}\overline{z}$, $s(z) = \frac{1}{\overline{z}}$, $s(z) = e^{2i\pi/n}z$.

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