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Finite Rings with Complete Bipartite Zero-Divisor Graphs

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Abstract—We describe all finite associative rings with complete bipartite zero-divisor graphs.

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1. INTRODUCTION

In this paper we consider associative rings (not necessarily commutative or containing an identity).

Definition. The *zero-divisor graph of a ring* R is a graph whose vertices are nonzero divisors of the zero of the ring (one-sided and two-sided), where two distinct vertices x and y are connected by an edge if and only if $xy = 0$ or $yx = 0$.

As usual, we denote the zero-divisor graph of a ring R by $\Gamma(R)$.

The notion of a zero-divisor graph was first introduced in [1]. I. Beck has introduced this notion for a *commutative* ring, considering all elements of the ring as the graph vertices. This definition was modified in [2], namely, only nonzero zero-divisors were treated as the graph vertices. Later the notion of a zero-divisor graph was generalized for *noncommutative* rings (e.g., [3]).

One of the main research directions in this field is the description of rings with zero-divisor graphs that satisfy certain conditions. Papers [4–7] completely describe finite rings with planar zero-divisor graphs. In [8] one gives a complete list of finite rings with Eulerian zero-divisor graphs. In this paper we completely describe finite rings whose zero-divisor graphs are bipartite.

A graph G is called *bipartite* if its vertex set V can be divided into two disjoint nonempty sets V_1 and V_2 in such a way that every edge in G connects vertices from different subsets. If a bipartite graph G has no multiple edges, and each vertex in V_1 is connected with each vertex in V_2, then G is called a *complete bipartite graph* ([9], P. 5). Complete bipartite graphs are denoted by $K_{n,m}$, where $n = |V_1|$ and $m = |V_2|$. The graph $K_{1,m}$ is called a *star*.

Earlier, in [2] one has described commutative rings with identity whose zero-divisor graphs are stars. In [4] one has obtained some results for commutative rings with identity and without nonzero nilpotents that have complete bipartite zero-divisor graphs.

Let us introduce some notions and denotations that are used in what follows.

Let an additive group of a ring R be decomposable into a direct sum of its nonzero additive subgroups A_i, where $i = 1, \dots, n$ and $n \geq 2$. Then $R = A_1 \dotplus \cdots \dotplus A_n$. If all subgroups are two-sided ideals of the ring R, then R is called *decomposable*, and the decomposition is written as $R = A_1 \oplus \cdots \oplus A_n$. Accordingly, a ring is called *indecomposable* if it is not decomposable.

Let $R = A_1 \oplus \cdots \oplus A_n$, $n \geq 2$. Since A_i is a two-sided ideal of the ring R, $i = 1, 2, \dots, n$, it follows that $A_i A_j \subseteq A_i \bigcap A_j = \{0\}$ whenever $i \neq j$, i.e., $A_i A_j = (0)$ whenever $i \neq j$. Conversely, let $R = A_1 \dotplus \cdots \dotplus A_n$ (a direct sum of additive subgroups) and $A_i A_j = (0)$ whenever $i \neq j$. Then all A_i are two-sided ideals of R.

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