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An Analog of the Cook Theorem for Polytopes

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Abstract—We prove that the polytope M of any combinatorial optimization problem with a linear objective function is an affine image of some facet of the cut polytope whose dimension is polynomial with respect to the dimension of M.

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Even in the middle of the last century it was known that in many combinatorial optimization problems the objective function is linear (in the sense described below). In what follows we call such problems linear combinatorial optimization problems (LCOP).

The linear combinatorial optimization problem LCOP_n**.** Assume that in a given set $E = \{e_1, e_2, \dots, e_n\}$, each element $e_i \in E$, $1 \le i \le n$, is associated with some weight $c_i = c(e_i) \in \mathbb{R}$, and $f : 2^E \to \{0, 1\}$ is a polynomially calculable (with respect to the number of elements n) rule. Let $S = \{s \subseteq E : f(s) = 1\}$ be the set of all feasible solutions of the problem. We seek for a subset $s \in S$ with the maximal (minimal) summary weight of elements.

Thus, for example, in the travelling salesman problem the set E is the set of paths connecting cities, and $c_i = c(e_i)$ are their lengths. The rule f in this problem returns 1 for routes going through all cities once. Many other optimization problems on graphs, including the cut problem, are stated analogously.

In the knapsack problem the symbol E denotes the collection of items which can be placed in the knapsack, $c_i = c(e_i)$ are their prices. The rule f in this problem returns 1, if the total size of the placed items does not exceed the size of the knapsack. One can interpret any other Boolean programming problem analogously.

The above “universal” statement allows us to introduce a “universal” notion of a polytope associated with an LCOP. For each feasible solution $s \in S$ we consider its characteristic vector $x = (x_i) \in \mathbb{R}^n$, $1 \le i \le n$, defined as follows:

$$x_i = \begin{cases} 1, & \text{if } e_i \in s; \\ 0 & \text{otherwise.} \end{cases}$$

As a rule, one understands the polytope $P(\mathrm{LCOP}_n)$ of the LCOP as the convex hull of the set of all such vectors. Therefore, $P(\mathrm{LCOP}_n)$ is a 0/1-polytope in $\mathbb{R}^n$. Note also that for each 0/1-polytope one can state the corresponding problem LCOP_n, neglecting the requirement of the polynomial calculability of the rule f.

Evidently, there exist combinatorial optimization problems which are not reducible to such a statement (in particular, due to the nonlinearity of the objective function). Nevertheless, even in such cases, introducing new variables, one can reduce such problems to the form LCOP_n. This allows one to use well-developed methods of the linear programming theory. Consider the next problem as a typical example.

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