Rigid solvable groups. Model theory

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The talk is based on the papers of the author and joint papers of A. Miasnikov and the author, which were published in the period from 2007 to the present. I also note the author's survey in the next monograph.

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Contents

Let the group G have an abelian normal subgroup A. The action of the group G on A by conjugations $a \to a^g = g^{-1}ag$ defines on A the structure of a right module over the group ring $\mathbb{Z}[G/A]$. The element

$$u = \alpha_1 \cdot g_1A + \ldots + \alpha_n \cdot g_nA \in \mathbb{Z}[G/A]$$

acts on $a \in A$ by the formula $a^u = (a^{g_1})^{\alpha_1} \cdot \ldots \cdot (a^{g_n})^{\alpha_n}$.

DEFINITION 1. *A group G is called m-rigid if it has a normal series*

$$G = \rho_1(G) > \rho_2(G) > \ldots > \rho_m(G) > \rho_{m+1}(G) = 1 \qquad (1)$$

with abelian factors $\rho_i(G)/\rho_{i+1}(G)$, each of which is a torsion-free $\mathbb{Z}[G/\rho_i(G)]$-module. That is, every nonzero element of the ring $\mathbb{Z}[G/\rho_i(G)]$ acts nontrivially on any nonzero element of the module $\rho_i(G)/\rho_{i+1}(G)$. The group G is called rigid if it is m-rigid for some m.

It follows from the definition and known facts that each ring $\mathbb{Z}[G/\rho_i(G)]$ is a (right and left) Ore domain and therefore is embedded into the division ring (skew field) of fractions (right and left coincide) which we denote by $Q(G/\rho_i(G))$. It can be proved that the series (1), if it exists, is uniquely determined by the group. Therefore we call it *a rigid series* and for its members we use the notation $\rho_i(G)$.

DEFINITION 2. *A rigid group G with the corresponding rigid series (1) is said to be divisible if each factor $\rho_i(G)/\rho_{i+1}(G)$ is a divisible module over the ring $\mathbb{Z}[G/\rho_i(G)]$. In other words, this means that $\rho_i(G)/\rho_{i+1}(G)$ will be a vector space over the division ring $Q(G/\rho_i(G))$.*

Any subgroup H of the rigid group G is also a rigid group. Its rigid series is obtained by intersecting H with the rigid series of G and discarding repetitions.

DEFINITION 3. *Call a subgroup H of a rigid group G independent (or an embedding of H into G independent) if for any i the linear independence of the system of elements of the module $\rho_i(G) \cap H/\rho_{i+1}(G) \cap H$, which is embedded in $\rho_i(G)/\rho_{i+1}(G)$, over the corresponding ring $\mathbb{Z}[H/\rho_i(G) \cap H]$ implies the linear independence of this system over the ring $\mathbb{Z}[G/\rho_i(G)]$ which contains $\mathbb{Z}[H/\rho_i(G) \cap H]$.*

We will see later that the notion of subgroup independence plays an extremely important role in the theory of rigid groups.

DEFINITION 4. A *m-rigid group G is called split if it splits into a sequential semidirect product $G_1 G_2 \dots G_m$ of abelian subgroups G_i. This means that G_i normalizes G_j for $i < j$, $\rho_i(G) = G_i \dots G_m$ and $G_i \cong \rho_i(G)/\rho_{i+1}(G)$.*

First, note that 1-rigid groups are Abelian groups that, like $\mathbb{Z}$-modules, are torsion-free; that is, they are simply torsion-free Abelian groups, and divisible 1-rigid groups are divisible Abelian groups without torsion i.e. they are direct sums of copies of the additive group of rational numbers.

EXAMPLE 1. The iterated wreath product $W = A_m \wr (A_{m-1} \wr (\ldots (A_2 \wr A_1) \ldots)$ with nontrivial torsion-free abelian groups A_i is a split m-rigid group.

EXAMPLE 2. A free solvable group of derived length m is an m-rigid group, its rigid series coincides with a derived series.

EXAMPLE 3. Let $(\alpha_1, \ldots, \alpha_m)$ be a finite tuple of nonzero cardinal numbers. We define a divisible split m-rigid group $M(\alpha_1, \ldots, \alpha_m)$ by induction on m. $M(\alpha_1)$ is the direct sum of α_1 copies of the additive group of rational numbers. Consider the group $B = M(\alpha_1, \ldots, \alpha_{m-1})$. It is, by induction, a divisible split $(m-1)$-rigid group. We embed the integer group ring $\mathbb{Z}B$ of this group into the division ring of fractions $Q(B)$. Let T be a right vector space over $Q(B)$ of dimension α_m. Then put $M(\alpha_1, \ldots, \alpha_m)$ equal to the matrix group $M = \begin{pmatrix} B & 0 \\ T & 1 \end{pmatrix}$. Otherwise, M is an extension of the additive group of the space T by the group B. Let B decomposes by induction into a semidirect product $M_1 \ldots M_{m-1}$ of abelian subgroups, with

$$M_1 \ldots M_i = M(\alpha_1, \ldots, \alpha_i), \ M_{i+1} \ldots M_{m-1} = \rho_{i+1}(B).$$

We set $M_m = \rho_m(M) = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$. The corresponding expansion for M is $M_1 \ldots M_{m-1} M_m$. It easily follows from the construction and induction that M is a divisible split m-rigid group.

The group $M(\alpha_1, \ldots, \alpha_m)$ is countable when all $\alpha_i \leq \aleph_0$. If this group is uncountable, then its cardinality coincides with the maximal cardinal α_i.

THEOREM (2008). 1) *Any divisible m-rigid group is split and isomorphic to a suitable group* $M(\alpha_1, \ldots, \alpha_m)$.
2) *Any m-rigid group can be independently embedded in a suitable group* $M(\alpha_1, \ldots, \alpha_m)$.

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Algebraic geometry over groups.

For a given group G, we call a G-group any group containing G as a fixed subgroup. A category of G-groups arises, in which morphisms are homomorphisms that act identically on G. Denote by $F = G * \langle x_1, \ldots, x_n \rangle$ the free product of the group G and the free group with basis $\{x_1, \ldots, x_n\}$. An expression of the form $v(x) = 1$, where $x = (x_1, \ldots, x_n)$ and $v(x) \in F$, is called an equation over G. Sometimes the element $v(x)$ itself is understood as an equation. The variables $x_1, \ldots, x_n$ are assumed to take valuations in G. The set $S \subseteq G^n$ of all solutions of some system of equations $\{v_i(x) = 1 \mid i \in I\}$ is called an algebraic subset in the affine space G^n. Let $\Theta(S) = \{v(x) \in F \mid v(s) = 1,\ s \in S\}$ be the annihilator of the nonempty algebraic set S. The coordinate group of this set is the quotient group $F/\Theta(S) = \Gamma(S)$. The group G is embedded in $\Gamma(S)$ which, as a G-group, is generated by the images of the elements $x_1, \ldots, x_n$.

The group F can be considered as a group of equations in x with coefficients from G. More generally, any group D that is generated by its subgroup G, and the set of elements $\{x_1, \ldots, x_n\}$, is called a group of equations over G if it satisfies the condition: any mapping $x \to (a_1, \ldots, a_n) \in G^n$ defines a G-epimorphism $D \to G$. Figuratively speaking, in this group, the variables $x_1, \ldots, x_n$ can be assigned any valuations from G. The group D is represented as a quotient group F/H. Among such groups D there is a group with maximal H equal to $\Theta(G^n)$, it is $\Gamma(G^n)$, in general D covers $\Gamma(G^n)$. The meaning of this concept is as follows. It often happens that it is convenient to consider as equations not elements of the group F and not elements of the group $\Gamma(G^n)$, last one sometimes difficult to describe, but elements of some intermediate group D.

Let S be a non-empty algebraic subset of G^n. If in the above definition of the group of equations D we restrict ourselves only to the mappings $x \to (a_1, \ldots, a_n) \in S$, then we obtain the definition of a group of equations on S, or, in other words, a group of equations over G under the condition $x \in S$. Such a group covers $\Gamma(S)$. Let D be a group of equations over G under the condition $x \in S$. Any G-epimorphism $D \to G$, defined by the mapping $x \to (a_1, \ldots, a_n) \in S$ is called a specialization, and the image of an element $v(x_1, \ldots, x_n)$ from D is denoted by $v(a_1, \ldots, a_n)$.

The Zariski topology is defined on the affine space G^n: all algebraic sets are taken as a prebase of a family of closed sets. If this topology is Noetherian, then any non-empty closed set can be uniquely represented as an uncancelled union of a finite number of irreducible components. A group G is called equationally Noetherian if, for any n, any system of equations in n variables over the group G is equivalent to one of its finite subsystems. The equationally Noetherian property for a group is equivalent to the Noetherian property of the Zariski topology on G^n for all n.

It is difficult to study the Zariski topology without this condition.

Principal for algebraic geometry over rigid groups result:

THEOREM (2009). *Any rigid group is equationally Noetherian*

First, let's talk about three new methods used in the proof.
1. Along with the group equations over a given group G we also consider *ring equations*. Let D be some group of equations over G in variables $x_1, \ldots, x_n$. Take the group ring $\mathbb{Z}D$ and an equation of the form $\gamma(x_1, \ldots, x_n) = 0$, where $0 \neq \gamma \in \mathbb{Z}D$ and the values of the variables $x_1, \ldots, x_n$ are still taken from G. If the equation under consideration has a solution, then the element γ, by necessity, lies in the augmentation ideal $(D-1)\mathbb{Z}D$. In this case, we represent γ as $u_1 + \ldots + u_m - v_1 - \ldots - v_m$, where $u_i, v_i \in D$ and the sets $\{u_1, \ldots, u_m\}$, $\{v_1, \ldots, v_m\}$ do not overlap. This representation is unique up to a permutation of u_i and v_i. Then the equation $\gamma(x_1, \ldots, x_n) = 0$ is equivalent to a disjunction of a finite number of finite systems of group equations

$$\bigvee_{\sigma \in S_m} (u_1 = v_{\sigma(1)} \wedge \ldots \wedge u_m = v_{\sigma(m)}), \tag{2}$$

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2. Let an m-rigid group G be represented as a semidirect product of abelian subgroups: $G = G_1 \ldots G_m$. In this situation, instead of a set of ordinary variables $(x_1, \ldots, x_n)$, it is convenient to consider m sets of *special variables* $(x_{1j}, \ldots, x_{nj})$, $j = 1, \ldots, m$, with the condition x_{ij} takes values in G_j. Let X denote the union of all such sets. Ordinary variables are expressed through special ones: $x_i = x_{i1} \ldots x_{im}$. On the contrary, special variables can be understood as ordinary ones, only satisfying the additional equations $[x_{ij}, g_j] = 1$, where $g_1, \ldots, g_m$ are fixed nontrivial elements from $G_1, \ldots, G_m$ respectively. Of course, the set of special variables X is associated with the choice of the decomposition $G = G_1 \ldots G_m$. However, it can be argued that for example the equationally Noetherian property of the group G in ordinary variables is equivalent to the Noetherian property by equations in special variables.

3. The third point is more complicated. We will consider the situation when the m-rigid group C is represented as a semidirect product $C = BA$, $A \lhd C$. The group A will be known to be residually torsion-free nilpotent. There is a natural B-epimorphism of the groups $C \to B$ with kernel A. It defines an epimorphism of the group rings $\mathbb{Z}C \to \mathbb{Z}B$. Take the corresponding division rings of fractions $Q(C)$ and $Q(B)$. We will need to lift the considered $\mathbb{Z}B$-epimorphism of the rings $\mathbb{Z}C \to \mathbb{Z}B$ to the $Q(B)$-epimorphism $R \to Q(B)$, where R is a sufficiently large subring of $Q(C)$ containing all fractions uv^{-1}, $v^{-1}u$ over $\mathbb{Z}C$ such that $v \notin (A-1)\mathbb{Z}C$. This can be done in the following way. First, a valuation is constructed on the ring $\mathbb{Z}A$, that is, the mapping $\omega : \mathbb{Z}A \longrightarrow \{0, 1, 2, \ldots, \infty\}$ with the conditions

$$\omega(u) = \infty \Leftrightarrow u = 0,\ \omega(uv) = \omega(u) + \omega(v),\ \omega(u+v) \geq \min\{\omega(u), \omega(v)\}.$$

For this valuation, the condition $\omega(u) \geq 1 \Leftrightarrow u \in (A-1)\mathbb{Z}A$ will also be satisfied. Then the valuation is extended to the ring $\mathbb{Z}C$ by the formula

$$\omega(b_1 u_1 + \ldots + b_s u_s) = \min\{\omega(u_1), \ldots, \omega(u_s)\},$$

here, $b_1, \ldots, b_s$ are various elements of B, $0 \neq u_i \in \mathbb{Z}A$. For $u \in \mathbb{Z}C$ we have $\omega(u) \geq 1 \Leftrightarrow u \in (A-1)\mathbb{Z}C$. Further, the valuation is extended to the division ring $Q(C)$ with values in $\mathbb{Z} \cup \{\infty\}$ if we set

$$\omega(uv^{-1}) = \omega(u) - \omega(v),\ 0 \neq u, v \in \mathbb{Z}C.$$

For a given $n \in \mathbb{Z}$ we put

$$Q_n = \{f \in Q(C) \mid \omega(f) \geq n\}.$$

It follows from the definition that $Q_m \cdot Q_n \subseteq Q_{m+n}$, Q_n is an additive subgroup in $Q(C)$, Q_0 is a subring, and Q_1 is an ideal in Q_0. Note also that $Q(B)$ is contained in Q_0 and $Q_0' = Q(B) + Q_1$ will be a subring in Q_0.

LIFTING LEMMA (2009). *The epimorphism $\mathbb{Z}C \to \mathbb{Z}B$ lifts to the epimorphism $Q_0' \to Q(B)$ with kernel Q_1. Any right fraction uv^{-1} (or left fraction $v^{-1}u$) over $\mathbb{Z}C$ such that $v \notin (A-1)\mathbb{Z}C$ which is equivalent to $v \notin Q_1$, lies in Q_0'.*

Of course, in the above scheme, much requires justification.

I will not list all the results on algebraic geometry over rigid groups and will only mention two results: we got the description of coordinate groups of irreducible algebraic sets over divisible rigid groups (2009) and found a reasonable formulation of Hilbert's Nullstellensatz for rigid groups and proved it (2015).

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In many ways, the theory of divisible *m*-rigid groups turned out to be similar to the classical theory of algebraically closed fields.

PROPOSITION (2017). *There is a recursive system of axioms in the standard group signature defining the class of all divisible m-rigid groups.*

PROPOSITION (2017). *Let $M = M(\alpha_1, \ldots, \alpha_m)$ be a countable divisible m-rigid group. There is a recursive system of axioms in the signature of group theory with constants from M, which defines the class of all divisible m-rigid groups containing M as an independent subgroup.*

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THEOREM (2017). *The theories $\mathfrak{T}_m$ and $\mathfrak{T}_m(M)$ are complete, so they are decidable and $\mathfrak{T}_m$ coincides with the elementary theory of any divisible m-rigid group and $\mathfrak{T}_m(M)$ coincides with the elementary theory with constants from M of any divisible m-rigid group into which M is independently embedded.*
COROLLARY. *Let $G \leqslant H$ be divisible m-rigid groups (M-groups). Then the embedding of G in H is elementary only if it is independent.*
THEOREM (2018). *The theories $\mathfrak{T}_m$ and $\mathfrak{T}_m(M)$ are ω-stable.*
COROLLARY. *In the class of rigid groups, exactly divisible rigid groups are ω-stable.*

We characterize divisible m-rigid groups as algebraically closed objects in the class of all m-rigid groups.

REMARK. In the paper of A. Baudisch and J.S. Wilson (1992) the question was posed: is it true that every solvable ω-stable group G has a normal nilpotent subgroup N such that G/N is an extension of an Abelian group by a finite one? It is true for groups of finite Morley rank. The answer to this question is negative: divisible m-rigid groups with $m \geq 3$ are not as such.

Recall that if the group $M(\alpha_1, \ldots, \alpha_m)$ is uncountable, then its cardinality coincides with the maximum α_i.

THEOREM (2018). *Let λ be an infinite cardinal number.*
1) *The group $M(\beta_1, \ldots, \beta_m)$ is λ-saturated if and only if $\lambda \leq \beta_i$ for all indices.*
2) *The group $M(\beta_1, \ldots, \beta_m)$ is saturated if and only if $\beta_1 = \ldots = \beta_m$ is an infinite cardinal.*
3) *A countable model of the theory $\mathfrak{T}_m(M)$ is saturated if and only if its corank over M is $(\aleph_0, \ldots, \aleph_0) = \aleph_0^m$.*
4) *Let $\lambda > \aleph_0$. The model of the theory $\mathfrak{T}_m(M)$ of the cardinality λ is saturated if and only if it has the form $M(\lambda, \ldots, \lambda) = M(\lambda^m)$.*

An important role is played by the divisible m-rigid group $M(\aleph_0, \ldots, \aleph_0) = M(\aleph_0^m)$, a countable saturated model of the theory $\mathfrak{T}_m$. It is proved that it will be the limit group of the Fraisse system of all finitely generated m-rigid groups. Let's give definitions adapted to our situation. For a given m-rigid group G, we denote by $\mathrm{age}(G)$ the set of all finitely generated independent subgroups of derived length m and by $\overline{\mathrm{age}}(G)$ the corresponding class of groups. Let $\mathcal{K}_m$ denote the class of all finitely generated m-rigid groups. We know that every finitely generated m-rigid group is independently embedded in a divisible m-rigid group of finite rank and hence, in the group $M(\aleph_0^m)$. Therefore, $\overline{\mathrm{age}}(M(\aleph_0^m)) = \mathcal{K}_m$. Let's call an m-rigid group G the limit for the class $\mathcal{K}_m$ if it satisfies the following properties:

(i) countable;

(ii) $\overline{\mathrm{age}}(G) = \mathcal{K}_m$;

(iii) homogeneity: if $U, V \in \mathrm{age}(G)$ and $\varphi : U \to V$ is an isomorphism, then it can be extended to an automorphism of G.

THEOREM (2018). *The limit group for the class $\mathcal{K}_m$ is uniquely determined and is isomorphic to $M(\aleph_0^m)$.*

Intersections of elementary submodels in models of theories $\mathfrak{T}_m$ and $\mathfrak{T}_m(M)$ are also studied.

THEOREM (2018). 1) *The intersection of some set of elementary submodels of a model of the theory $\mathfrak{T}_m$ is an elementary submodel if and only if its derived length equal to m.*
2) *The intersection of any set of elementary submodels of the theory $\mathfrak{T}_m(M)$ is again an elementary submodel.*

The next theorem concerns the quantifier elimination of the theories under study.

THEOREM (2018). *Any formula in the theory $\mathfrak{T}_m$ or in the theory $\mathfrak{T}_m(M)$ is equivalent to a Boolean combination of $\forall\exists$-formulas.*

Recall that the (complete) *type* of a finite tuple $\bar{a} = (a_1, \ldots, a_n)$ of elements of the group G is the set of all formulas in n free variables in the signature of group theory that are true on this tuple. Let's denote this type by $tp(a_1, \ldots, a_n)$. A group G is called *strongly* $\aleph_0$*-homogeneous* if the equality $tp(a_1, \ldots, a_n) = tp(b_1, \ldots, b_n)$ implies that the tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are conjugate by an automorphism of the group. It was proved that an arbitrary free group has this property. Let us also introduce the ∃-type of the tuple $tp_\exists(a_1, \ldots, a_n)$, that is, the set of ∃-formulas that are true on the given tuple.

THEOREM (2018). *Let G be a divisible rigid group. Then $tp_\exists(a_1, \ldots, a_n) = tp_\exists(b_1, \ldots, b_n) \Leftrightarrow$ the tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are conjugate by an automorphism of G.*

COROLLARY 1. *Let G be a divisible rigid group. Then the complete type of any finite tuple of elements of G is determined by its $\exists$-fragment.*
COROLLARY 2. *The divisible rigid group is strongly $\aleph_0$-homogeneous.*
COROLLARY 3. *For the theory $\mathfrak{T}_m$, the statement of Theorem 21 can be strengthened: the quantifier elimination to a Boolean combination of $\exists$-formulas can be done.*

Corollaries 1 and 2 are obtained from the theorem in an obvious way, and Corollary 3 is deduced from Corollary 1 and Theorem of Bruno Poizat, which asserts that if the complete types in the models of given theory are determined by formulas of some set F, then quantifier elimination to a Boolean combination of F-formulas can be done. Note that the theory $\mathfrak{T}_m$ for $m > 1$ is not model complete, and thus it cannot quantifier elimination to $\exists$-formulas (or $\forall$-formulas), so the bound we obtained for quantifier elimination is fairly accurate.

We shall now describe definable subgroups in divisible rigid groups.

THEOREM (2020). *Members of the rigid series of a divisible rigid group and no others are definable without parameters subgroups in the signature of the group theory.*

We remember that a divisible m-rigid group G can be split into a semidirect product $G_1 G_2 \ldots G_m$ of abelian subgroups. The subgroup G_i from this splitting coincides with the centralizer of its arbitrary nontrivial element g_i. Let us fix the set of indices $(i_1, i_2, \ldots, i_k)$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq m$, and take the subgroup $G_{i_1} G_{i_2} \ldots G_{i_k}$. It will be divisible and is determined by the formula with parameters of the free variable x:

$$\exists x_1 \ldots \exists x_k(([x_1, g_{i_1}] = 1) \wedge \ldots \wedge ([x_k, g_{i_k}] = 1) \wedge (x = x_1 x_2 \ldots x_k)).$$

THEOREM (2020). *A subgroup of a divisible m-rigid group G is definable with parameters from G in the signature of the group theory if and only if it has the form* $G_{i_1} G_{i_2} \ldots G_{i_k}$ *for some splitting* $G_1 G_2 \ldots G_m$ *of the group G into a semidirect product of abelian subgroups.*

REMARK. It is clear that in the non-Abelian case there will be an infinite number of subgroups definable with parameters. But the subgroups corresponding to the same set of indices $(i_1, i_2, \ldots, i_k)$ are conjugate. Therefore, up to conjugacy, there will be exactly 2^m subgroups definable with parameters.

Let $G = M(\alpha_1, \ldots, \alpha_m)$ and the cardinal λ strictly greater than $\aleph_0$ and all α_i. We will assume that the group G is elementarily embedded in the group $\mathbb{G} = M(\lambda, \ldots, \lambda)$. The latter is understood as a large (monster) model for G. It is known that there are no proper subgroups of finite index in G. Then we can assert that G has only one generic type - the type $p \in S_1(G)$, whose Morley rank $\mathrm{RM}(p)$ coincides with the Morley rank $\mathrm{RM}(G)$ of the group G. Accordingly, the element $x \in \mathbb{G}$ with the condition $\mathrm{tp}(x/G)$ is a generic type, is called a generic element. An algebraic description of generic elements will be given below, but one more definition is needed first.

For any splitting $G_1 \dots G_m$ of the group G, there is a unique compatible splitting $\mathbb{G}_1 \dots \mathbb{G}_m$ of the group $\mathbb{G}$. In it, $\mathbb{G}_i$ equals the centralizer in $\mathbb{G}$ of the nontrivial element $g_i \in G_i$.

DEFINITION *An element $x \in \mathbb{G}$ is said to be independent with G if for some (any) splitting $G_1 \dots G_m$ of the group G and compatible with it splitting $\mathbb{G}_1 \dots \mathbb{G}_m$ of the group $\mathbb{G}$ and the corresponding splitting $x_1 \dots x_m$ of the element x and any i the following condition is satisfied:*
the element x_i together with any system of elements from G_i linearly independent over $\mathbb{Z}[G_1 \dots G_{i-1}]$ constitutes a linearly independent system over $\mathbb{Z}[\mathbb{G}_1 \dots \mathbb{G}_{i-1}]$.
THEOREM. *An element of $\mathbb{G}$ is generic over G if and only if it is independent with G.*

Now we consider the problem of calculating Morley rank.

THEOREM. *Let G be a countable saturated model of the theory $\mathfrak{T}_2$ of divisible metabelian rigid groups. Then the Morley rank of G equal to $\omega + 1$.*

Let us fix a splitting $G_1 G_2 \ldots G_m$ of a divisible m-rigid group G. The subgroup G_i from this splitting is definable, since it coincides with the centralizer of its arbitrary nontrivial element. Assign to each tuple $(n_1, \ldots, n_m)$ of nonnegative integers an ordinal

$$\alpha = \omega^{m-1} n_m + \ldots + \omega n_2 + n_1$$

and denote it by L_α set $G_1^{n_1} \times G_2^{n_2} \times \ldots \times G_m^{n_m}$, which is obviously definable over G in $G^{n_1 + \ldots + n_m}$. We have a plan to prove the following result.

HYPOTHESIS. *Let G be a countable saturated model of the theory $\mathfrak{T}_m$ and fixed its splitting $G_1 G_2 \ldots G_m$ into a semidirect product of abelian subgroups. Then the Morley rank of the set*

$$L_\alpha = G_1^{n_1} \times G_2^{n_2} \times \ldots \times G_m^{n_m}$$

relative to G is α.

COROLLARY. $\mathrm{RM}(G) = \omega^{m-1} + \omega^{m-2} + \ldots + 1$.

Using one reasoning as an example, we will show how algebraic geometry is used in the theory of models of rigid groups.
Let G be a divisible rigid group and consider a definable subset of G^n of Morley rank $\alpha > 0$. It can be assumed that the corresponding formula with parameters in G has the form: $\Phi(Y) = \exists X \Psi(X, Y)$, where Ψ is a Boolean combination without negations of equalities of the form $f = 1$, inequalities of the form $f \neq 1$ and expressions $\neg P_{j,s}(f_1, \ldots, f_s)$, $f, f_1, \ldots, f_s$ are terms. Let $\Psi_1 \vee \ldots \vee \Psi_r$ be the disjunctive normal form of Ψ and $\Phi_i(Y) = \exists X \Psi_i(X, Y)$. We can assert that the Morley rank of one of the sets $\Phi_i(G)$ is equal to α, as a result, the problem is reduced to the case when Ψ is a conjunction of given expressions. All equalities in Ψ define an algebraic subset S in the corresponding affine space.

We rewrite the formula as

$$\Phi(Y) = \Phi_S(Y) = \exists X(((X, Y) \in S) \wedge \Omega(X, Y)),$$

here Ω is obtained by eliminating the equalities from Ψ. If in S there is a proper closed subset P such that the Morley rank of the set $\Phi_P(G)$ is equal to α, then we replace S by P. Since the Zarisski topology is Noetherian, we will assume that S is minimal, that is, for any proper closed subset P, the Morley rank of the set $\Phi_P(G)$ is less than α. We show that the set S is irreducible. After that the terms in the formula Φ are identified with the elements of the coordinate group $\Gamma(S)$. If $1 \neq f \in \Gamma(S)$, then the equality $f = 1$ defines in S a proper closed subset, its Morley rank $< \alpha$, and then, when the inequality $f \neq 1$ is added to Ω, the Morley rank of the set $\Phi(G)$ remains equal to α. After that we use the description of the coordinate groups of irreducible algebraic sets over G. We also use Lifting Lemma.

THANK YOU !