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REPRESENTATION OF UNIVALENT FUNCTIONS AS INFINITE COMPOSITIONS

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Introduction

The problem of representation of holomorphic univalent functions as an infinite composition started to attract attention rather recently. Thus, for example, [1] and [2] are devoted to representation of functions which map the unit disk $\mathbb{U} = \{z : |z| < 1\}$ onto hyperbolically convex triangles with interior angles equal to $\pi/2^n$ and 0, and of the functions of the Schottky class in the form $f = \cdots \circ k_{\alpha_n}^{\gamma_n} \circ \cdots \circ k_{\alpha_1}^{\gamma_1}$, where $k_\alpha^\gamma(z) = e^{i\gamma} k_\alpha(e^{-i\gamma} z)$, $0 < \alpha \leqslant 1$, $\gamma \in \mathbb{R}$ and

$$k_\alpha(z) = \frac{2\alpha z}{1 - z + \sqrt{(1-z)^2 + 4\alpha^2 z}}, \quad z \in \mathbb{U}.$$

The function $k_\alpha(z)$ maps $\mathbb{U}$ onto hyperbolic digon. In [3] a similar representation was given for the logarithm and some elliptic integrals of the first kind and also for functions mapping $\mathbb{U}$ onto hyperbolically convex polygons with interior angles equal to $\pi/2^n$ or 0.

In this article we use the new approach for representation of univalent functions in the form of infinite compositions. We introduce the notation. Let $\mathcal{M}$ stand for the class of holomorphic functions which map univalently $\mathbb{U}$ into itself with the norming $f(0) = 0$ and $f'(0) > 0$, and $\mathcal{P}$ be the class of functions p holomorphic in $\mathbb{U}$, $\operatorname{Re} p(z) > 0$, $z \in \mathbb{U}$, $p(0) = 1$. We denote by S the class of holomorphic univalent functions in $\mathbb{U}$ with the norming $f(0) = f'(0) - 1 = 0$.

Let us consider the Loewner differential equation

$$\frac{dw}{dt} = -w \frac{1 - \xi(t) w}{1 + \xi(t) w}, \quad t \geqslant 0, \quad w|_{t=0} = z, \quad z \in \mathbb{U}, \tag{1}$$

where $\xi(t)$ is a piecewise continuous function, $|\xi(t)| = 1$, which will be called *control* in what follows. The integrals of the Loewner equation (1) form a subclass dense everywhere in the class $\mathcal{M}$. In particular, if $\xi(t) = \text{const}$, then for every $t > 0$ the integral $w(z,t)$ of the differential equation (1) is a function p_α^γ, $e^{-t} = \alpha$, $e^{i\gamma} = \xi$, which maps $\mathbb{U}$ onto the unit disk with radial cut with end point $e^{i\gamma}$. The solution of the Loewner equation (1) with a piecewise constant control is the composition of the functions p_α^γ.

Let us consider a sequence of piecewise constant functions $\xi_n(t)$, which converges uniformly to $\xi(t)$. Since $\frac{1-\xi w}{1+\xi w}$ depends continuously on ξ, we have that the sequence of functions $\frac{1-\xi_n(t)w}{1+\xi_n(t)w}$ converges weakly with respect to t to the function $\frac{1-\xi(t)w}{1+\xi(t)w}$, and, consequently, the integrals $w_n(z,t)$ of the Loewner equation (1) with the controls ξ_n for every $t > 0$ converge uniformly to the integral $w(z,t)$ with the control ξ (see [4]). From the fact that the class of piecewise constant controls ξ is

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