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THE BESSEL FUNCTION ON A FINITE FIELD

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Analogs of Bessel functions with the argument in a finite field arise in both the theory of special functions (see [1], [2]) and theory of representations of groups (see [3]–[6]). In this article we study an analog of the Bessel function via which the matrix elements of irreducible representations of the group of motions of plane over a finite field are expressed. The method developed in [7] allows us to obtain the Hansen formula, the theorems of addition and multiplication.

1. Definition of the Bessel function

Let $\mathbb{F}$ be a finite field composed of $q = p^n$ elements (p is the characteristic of the field), $\mathbb{F}^*$ the multiplicative group of the field, $\mathbb{F}(\tau) = \{z\}$ the square expansion with the conjugation $\overline{z}$ and the norm $N(z) = z\overline{z}$. We denote by η the generator of the cyclic group $\mathbb{F}^*(\tau)$. For each $u \in \mathbb{F}^*$, the circle $C_u = \{z \in \mathbb{F}(\tau) \mid N(z) = u\}$ consists of $q+1$ points. The element $\xi = \eta^{q-1}$ generates a cyclic group $C \equiv C_1$ of unit of the field $\mathbb{F}(\tau)$ (see [8]). Let $k(t)$ be the exponent in the equality $t = \xi^{k(t)}$, $t \in C$. As is known (see, e. g., [8]), the characters of the group C have the form

$$\pi_s(t) = \exp\left(\frac{2\pi i}{q+1} s k(t)\right), \quad s = 0, 1, \dots, q,$$

where s is the number of the character. We introduce the bilinear form $\langle z_1, z_2\rangle = z_1 z_2 + \overline{z}_1\overline{z}_2$ and fix a nontrivial additive character χ of the field $\mathbb{F}$. The Bessel function $J_s(z)$ of the argument $z \in \mathbb{F}(\tau)$ and index $s = 0, 1, \dots, q$ with values in the field $\mathbb{C}$ of complex numbers is defined by the equality

$$J_s(z) := \frac{1}{q+1} \sum_{t, N(t)=1} \chi(\langle z, t\rangle)\pi_s(t).$$

A function close to $J_s(z)$ was considered in [4]

$$j_r(u) = \frac{1}{q} \sum_{t, N(t)=u} \chi(t + \overline{t})\varphi_r(t), \tag{1}$$

where $r = 0, 1, \dots, q^2 - 2$, $\varphi_r(t)$ is the multiplicative character of the field $\mathbb{F}^*(\tau)$. Let $[r] \equiv r(\operatorname{mod}(q+1))$ and $0 \le [r] \le q$; then the connection between the mentioned functions is as follows:

$$j_r(z\overline{z}) = \frac{q+1}{q}\varphi_r(z) J_{[r]}(z). \tag{2}$$