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Accuracy of the Penalty Method for Parabolic Variational Inequalities with an Obstacle Inside the Domain

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INTRODUCTION

The penalty methods play an important role in the modern theory of variational inequalities; they are significant both for the theory and practice. One uses these methods, proving the existence and the regularity of a problem solution (e.g., [1, 2]) and developing various numerical methods [3].

In this paper we consider parabolic variational inequalities with the one-sided constraint $u \geq 0$ in a domain with a null obstacle inside. The initial problem has the following form: find a function $u \in \mathcal{K} \cap \mathcal{W}$ such that

$$u(0) = u_0 \in H, \quad \int_0^T \langle u' + Au, v - u\rangle\, dt \geq \int_0^T (f, v - u)\, dt \;\; \forall v \in \mathcal{K}, \qquad (P)$$

where $\mathcal{K} = \{v \in \mathcal{V}^g : v \geq 0 \text{ a. e. in } Q\}$, $\mathcal{V}^g = \{v \in L_2(0, T; V) : v = g \text{ on } \Sigma\}$.

Here the operator A is generated by a strongly monotone vector field a,

$$\int_0^T \langle Au, v\rangle dt = \int_0^T \int_\Omega a(x, t, u_x) \cdot v_x\, dx\, dt \;\; \forall v \in L_2(0, T; V^0),$$

where $u_x = (u_{x_1}, \dots, u_{x_n})$, $u_{x_i} = \partial u/\partial x_i$. Assume that a satisfies the standard conditions, ensuring the unique solvability of Problem (P).

The penalty method reduces this problem to the operator equation

$$u_\varepsilon \in \mathcal{W} \cap \mathcal{V}^g : u'_\varepsilon + Au_\varepsilon + \beta_\varepsilon(u_\varepsilon) = f, \quad u_\varepsilon(0) = u_0,$$

with a small parameter $\varepsilon > 0$ and a certain penalty operator β_ε. One can define β_ε by several known formulas. Usually one puts

$$\beta_\varepsilon(u) = -\frac{1}{\varepsilon} u^-, \quad u^- = -\min(0, u).$$

The following accuracy estimates are typical (e.g., [4], [5], [6], P. 122):

$$\|u - u_\varepsilon\|_{L_\infty(Q)} \leq c\varepsilon \|f\|_{L_\infty(Q)}, \quad \|u - u_\varepsilon\|_{L_2(0,T;V)} \leq c\varepsilon^{1/2}. \qquad (1)$$

Hereinafter the constant c is independent of ε. The proof of the second inequality in (1) (in the energetic norm) is based on a priori estimates of certain norms $\beta_\varepsilon(u_\varepsilon)$; it requires that the problem is regular in a sense, namely, the initial data in inequality (P) are sufficiently smooth.

In this paper we study the accuracy of the penalty method with the operator

$$\beta_\varepsilon(u) = -f^- \chi_\varepsilon(u), \quad \chi_\varepsilon(u) = \frac{1}{\varepsilon}(u - \varepsilon)^- \qquad (2)$$

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