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SPECTRAL THEOREM IN A SPACE WITH BILINEAR FORM

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The spectral theorem plays the fundamental role in the functional analysis. It is intensively used, for example, in the construction of the quantum theory of measure and integral (see [1]). A generalization of this theorem to operators in a space with bilinear form (b. f.) which defines an indefinite metric is known (see [2], also [3], Chap. IV, p. 252). In this article the spectral theorem is expanded to operators which are self-adjoint with respect to bounded b. f., which are defined by boundedly inverses.

Let H be a complex Hilbert space with the scalar product (s. p.) $(\cdot,\cdot)$. We denote by $B(H)$ a set of all bounded operators in H and by G a set of all operators from $B(H)$, possessing a bounded inverse. Let $T \in B(H)$. We introduce the b. f. $\langle f, g\rangle_T := (Tf, g)\ \forall f, g \in H$. Sometimes, in a special case $T \in G$, $T > 0$, the bilinear form $\langle\cdot,\cdot\rangle_T$ will be denoted by $(\cdot,\cdot)_T$. Let us note that $(\cdot,\cdot)_T$ possesses all properties of an s. p., and the linear space H endowed with $(\cdot,\cdot)_T$ is a Hilbert space. It will be denoted by H_T.

The b. f. $a(\cdot,\cdot)$ is said to be *unitary generated* if a unitary with respect to $(\cdot,\cdot)$ operator Y exists such that $a(\cdot,\cdot) = (Y\cdot,\cdot)$. The operator $B^* \in B(H)$ is said to be *adjoint* to $B \in B(H)$ (= T-*adjoint*) with respect to the b. f. $\langle\cdot,\cdot\rangle_T$, if $\langle Bf, g\rangle_T = \langle f, B^*g\rangle_T\ \forall f, g \in H$. For unitary selfadjoint $J \in G$, $J \neq \pm I$ the b. f. $\langle\cdot,\cdot\rangle_J$ is usually denoted by $[\cdot,\cdot]_J$ and is called an indefinite metric. Everywhere in what follows we set that $U = \int_0^{2\pi} e^{i\lambda} e_U(d\lambda)$ is a unitary operator. Here $e_U : \mathcal{B}_S \to \Pi_U$ is a spectral measure on the σ-algebra $\mathcal{B}_S$ of Borel subsets of the unit sphere $S \subset C$ and Π_U is a set of all orthogonal projections in H.

Let us note the following obvious

Proposition 1. *Let $T \in G$. The operator B is T-selfadjoint if and only if $B = T^{-1}B^*T$ $(= T^{*-1}B^*T^*)$ or, which is the same, $TB = B^*T$.*

If B is T-selfadjoint, then $(T^*T^{-1})B^* = B^*(T^*T^{-1})$.

Let $T \in G$, $T = XY$, where $X > 0$ and $X, Y \in G$. The operator $A \in B(H)$ is T-selfadjoint in H if and only if A is Y-selfadjoint in H_X.

Both A and A^* are U-selfadjoint simultaneously and $U(AA^*) = (A^*A)U$, $U(A^*A) = (AA^*)U$.

Let us prove, for example, the last equality. Let A be U-selfadjoint. From the relation $A = U^{-1}A^*U = UA^*U^{-1}$ we have $UA = A^*U$ and $UA^* = AU$. Therefore

$$U(A^*A) = (UA^*)A = (AU)A = A(UA) = A(A^*U).$$

Let $T \in B(H)$, $T > 0$. If the operator $Y \in B(H)$ is such that $Y^{-1} = T^{-1}Y^*T$, then Y is a unitary operator in H_T.

The next proposition can be verified straightforwardly.

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