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PROBLEM OF DEDUCIBILITY OF IDENTITIES IN FINITE RINGS

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In [1] L.A. Bokut' stated the question on the solvability of the general problem of deducibility of identities for groups: "Whether an algorithm exists which along any system of group words $f_1, \dots, f_m$ (of a fixed set of variables $x_1, x_2, \dots$) and a single word f clarifies if the identity $f = 1$ follows from the identities $f_1 = 1, \dots, f_m = 1$?". The negative answer to this question was obtained in [2]. In [3] the unsolvability of the general problem of deducibility of identities was proved for groupoids. It was noted in the same paper that the general problem of deducibility of identities is unsolvable also for semigroups. In [4] the author noted that a ring identity exists for which no algorithm exists which decides along an arbitrary finitely based ring variety whether in this variety the identity is fulfilled or not. It seems natural to consider the question mentioned above for classes of finite semigroups, groups, and rings. A solution of the general problem of deducibility of an identity in the case of finite semigroups and groups is yet unknown. The result of this article is the negative answer for this problem in case of finite rings.

Theorem. *No algorithm exists which along an arbitrary finitely based ring variety could decide whether the class of all finite rings from this variety satisfies the identity* $(xy)(zt) = 0$.

Before we turn to the proof of the theorem let us obtain some auxiliary propositions. The proof of the theorem will be based on an interpretation of the action of two-tape Minsky's machine (see [5]). This method was first used by Yu.Sh. Gurevich in [6] in the proof of unsolvability of the quasi-equational theory of classes of finite semigroups and associative rings. In [7] a detailed exposition of methods for interpretation of Minsky's machines and a survey of results related to interpretation of these machines can be found.

Let P be a certain recursively enumerable nonrecursive set of natural numbers, $\mathfrak{p}$ a partial characteristical function of the set P (see [5]). We denote by M a two-tape Minsky's machine computing the function $\mathfrak{p}$ (see [8]). In what follows we will assume that $q_0, q_1, q_2, \dots, q_m$ are the interior states of the machine M, q_1 being the initial state and q_0 the final state. If the machine is in the state q_i and the j-th band is shifted by ξ_j cells to the left, then we will say that M is in the configuration $q_i\xi_1\xi_2$.

Before we pass to proper proof of the theorem, we introduce the notation. Let $\mathcal{K}$ be a certain (not obligatory associative) ring, $a_1, a_2, \dots, a_{n-1}, a_n \in \mathcal{K}$. We define the left-normed product $a_1a_2\dots a_{n-1}a_n$ by induction with respect to n, assuming that $a_1a_2\dots a_n = (a_1a_2\dots a_{n-1})a_n$ for $n > 1$. For $n = 1$, we have simply a_1. Further, we assume $a_1 \underbrace{a_2 \dots a_2}_{m \text{ times}} = a_1a_2^m$, b^0 is the empty symbol for any symbol b. Let $X = \{x_1, x_2, \dots, x_n, \dots\}$ be a certain countable set, Γ a free groupoid with the set of free generators X. We define the mappings A, B, C, D, E, F, G, H, K, which transform elements of the Cartesian product $\Gamma \times \Gamma$ into elements of the groupoid Γ, assuming

$$A(y, x) = (x(x(yx)x^7)x), \qquad E(y, x) = (x(x(yx)x^{11})x),$$