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Some Extremal Problems for Algebraic Polynomials in Loaded Spaces

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Abstract—In a loaded Jacobi space with the inner product

$$\langle f,g\rangle=\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}fg(1-x)^{\alpha}(1+x)^{\beta}dx+Lf(1)g(1)+Mf(-1)g(-1)\ \ (L,M\geq 0)$$

we consider the lth derivative of the algebraic polynomial $\Pi_N^{(r)}(x)=\sum\limits_{k=N-r+1}^{N}a_k^{(0)}x^k+\sum\limits_{j=0}^{N-r}a_jx^j$ ($a_N^{(0)}>0$) with fixed coefficients $a_k^{(0)}$. We solve the following extremal problems: Find $\inf\langle D^l[\Pi_N^{(r)}(x)],D^l[\Pi_N^{(r)}(x)]\rangle$ ($D=\frac{d}{dx}$, $0\leq l\leq N-r$) and calculate extremal polynomials.

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To the memory of P. L. Ul'yanov

1. INTRODUCTION

In the space of real polynomials $\mathbb{P}$ on the segment $[-1,1]$ we consider the following bilinear form:

$$\langle f,g\rangle=\int_{-1}^{1}f(x)g(x)d\mu(x)+\sum_{k=1}^{m}M_kf(x_k)g(x_k),\tag{1.1}$$

where $d\mu(x)$ is a probabilistic measure on $[-1,1]$, $M_k (k=1,2,\dots,m)$ are nonnegative numbers, and x_k $(k=1,2,\dots,m)$ are points from the segment $[-1,1]$. This bilinear form defines the inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients. The completion of $\mathbb{P}$ in the norm $\|f\|=\langle f,f\rangle^{\frac{1}{2}}$ leads to the corresponding loaded space of functions. Such spaces occur in some problems of mathematical physics, computing mathematics, the theory of functions and functional analysis (see, for example, [1–5], references therein, and papers [6–17]). In particular, spectral problems for ordinary differential equations, boundary value problems with spectral parameter in boundary conditions, differential equations with discontinuous coefficients, and loaded integral equations lead to such spaces.

By applying the Gram–Schmidt orthogonalization to the canonical basis of $\mathbb{P}$ we construct polynomials of degree n: $\widehat{q}_n(x) (n\in\mathbb{Z}_+; x\in[-1,1])$ orthonormal in the inner product (1.1):

$$\widehat{q}_n(x)=k_n^{(n)}x^n+k_{n-1}^{(n)}x^{n-1}+\dots+k_0^{(n)},\ \ k_n^{(n)}>0,\tag{1.2}$$

and

$$\langle\widehat{q}_n,\widehat{q}_s\rangle=\int_{-1}^{1}\widehat{q}_n(x)\widehat{q}_s(x)d\mu(x)+\sum_{k=1}^{m}M_k\widehat{q}_n(x_k)\widehat{q}_s(x_k)=\delta_{s,n}\ \ (s,n\in\mathbb{Z}_+).$$

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