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THE LOCAL STRUCTURE OF SURFACE OF INTERIOR CONFORMAL RADIUS FOR A PLANE DOMAIN

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An interior conformal radius $R(\mathcal{D},b)$ of a simply connected domain $\mathcal{D}$ on the complex plane z at a point $b\in\mathcal{D}$ represents the radius of a disk conformally equivalent to the domain $\mathcal{D}$ with strengthening of norming in the Riemann theorem. Namely, having denoted by $\zeta=\mathcal{F}_R(z)$ the mapping function, we have

$$\mathcal{F}_R(b)=0,\quad \mathcal{F}'_R(b)=1;\quad R(\mathcal{D},b)=|\mathcal{F}_R(t)|,\quad t\in\partial\mathcal{D}.$$

With the mapping $\zeta=\mathcal{F}(z)$ onto the unit disk $E=\{\zeta:|\zeta|<1\}$ with the usual norming $\mathcal{F}(b)=0$, $\arg\mathcal{F}'(b)=\beta$ one can write out $R(\mathcal{D},b)=1/|\mathcal{F}'(b)|$, because $\mathcal{F}_R(z)=\mathcal{F}(z)/\mathcal{F}'(b)$; therefore, $|\mathcal{F}_R(t)|=|\mathcal{F}(t)/\mathcal{F}'(b)|=1/|\mathcal{F}'(b)|$ for $t\in\partial\mathcal{D}$. Using the inverse function

$$z=\mathcal{F}^{-1}(\zeta)=f\left(\frac{\zeta+a}{1+\overline{a}\zeta}\right),$$

which maps the disk E onto the domain $\mathcal{D}$ so that $0\mapsto a\mapsto f(a)=b$, we obtain

$$R(\mathcal{D},b)=|\mathcal{F}'(b)|^{-1}=|(\mathcal{F}^{-1})'(0)|=|f'(a)|(1-|a|^2)=R(f(E),f(a));\quad b=f(a),\quad \mathcal{D}=f(E).$$

In this article we study the local behavior of the surface of interior conformal radius over a disk and over a domain $\mathcal{D}$ and deduce the conclusions about the structure of the surface of conformal radius $R(\mathcal{D},b)$ and the surface for the density of the hyperbolic metric $\lambda(\mathcal{D},b)=1/R(\mathcal{D},b)$ in some subclasses of the domains $\mathcal{D}$. Theorems 1–3 complement the known effective results in [1]–[3] about the global structure of the surface of conformal radius, whose essence is in the following characterizations:

1°. $\omega=R(\mathcal{D},z)$ is the equation for a convex upwards surface $(\infty\notin\mathcal{D})\Leftrightarrow\mathcal{D}$ is a convex domain;

2°. $\omega=R(\mathcal{D},z)$ is the equation of a convex downwards surface $(\infty\in\mathcal{D})\Leftrightarrow\overline{\mathbb{C}}\setminus\overline{\mathcal{D}}$ is a convex domain.

From a slightly modified substantiation of characterization 1° in [2], in Theorem 4 we obtain the equivalence of the three relations:

$$\operatorname{Re}\zeta\frac{f''(\zeta)}{f'(\zeta)}\geq -1,\quad \zeta\in E; \tag{1.1}$$

$$\frac{1}{2}(1-|\zeta|^2)^2|\{f(\zeta),\zeta\}|\leq 1-\frac{1}{4}\left|\frac{f''(\zeta)}{f'(\zeta)}(1-|\zeta|^2)-2\overline{\zeta}\right|^2,\quad \zeta\in E; \tag{1.2}$$

the surface with the equation $\omega=R(f(E),z)$ is convex upwards. (1.3)

The convexity upwards of the surface of conformal radius $\omega=R(f(E),f(\zeta))$ over a disk E results in the convexity of the domain $\mathcal{D}=f(E)$ (see Theorem 5); though the inverse assertion is not true.

1. In [4] the local behavior of the conformal radius of a simply connected domain $f(E)$ over the disk E was substantiated. Namely,

$$R(\mathcal{D}, z) = R(f(E), f(\zeta)) \stackrel{\text{def}}{=} R_1(\zeta) = R_1(a+\rho e^{i\theta}) = R_1(a)(1 + b_1\rho + b_2\rho^2 + O(\rho^3)), \qquad (2)$$

where

$$b_1 = \operatorname{Re} B_1, \quad B_1 = \left(\frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{1-|a|^2}\right)e^{i\theta}, \qquad (3)$$

$$b_2 = \frac{1}{2}\operatorname{Re}\left(\left[\frac{f'''(a)}{f'(a)} - \left(\frac{f''(a)}{f'(a)}\right)^2\right]e^{i2\theta}\right) + \\ + \frac{1}{2}\left[\operatorname{Re}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\right]^2 - \operatorname{Re}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\frac{2\operatorname{Re}(\overline{a}e^{i\theta})}{1-|a|^2} - \frac{1}{1-|a|^2}. \qquad (4)$$

Decomposition (2) can be reached by means of the representation in the disk E

$$R_1(\zeta) = R_1(a+\rho e^{i\theta}) = \\ = R_1(a) + \left(\frac{\partial R_1}{\partial\zeta}e^{i\theta} + \frac{\partial R_1}{\partial\overline{\zeta}}e^{-i\theta}\right)\bigg|_{\zeta=a}\rho + \frac{1}{2}\left(\frac{\partial^2 R_1}{\partial\zeta^2}e^{i2\theta} + 2\frac{\partial^2 R_1}{\partial\zeta\partial\overline{\zeta}} + \frac{\partial^2 R_1}{\partial\overline{\zeta}^2}e^{-i2\theta}\right)\bigg|_{\zeta=a}\rho^2 + \\ + O(\rho^3) = R_1(a) + 2\operatorname{Re}\left(\frac{\partial R_1}{\partial\zeta}(a)e^{i\theta}\right)\rho + \left[\operatorname{Re}\left(\frac{\partial^2 R_1}{\partial\zeta^2}(a)e^{i2\theta}\right) + \frac{\partial^2 R_1}{\partial\zeta\partial\overline{\zeta}}(a)\right]\rho^2 + O(\rho^3). \qquad (2')$$

From decomposition (2) we can easily obtain the representation for the conformal radius in the power (-1), i.e.,

$$\frac{1}{R_1(a+\rho e^{i\theta})} = \frac{1}{R_1(a)}(1 - b_1\rho + (b_1^2 - b_2)\rho^2 + O(\rho^3)) \stackrel{\text{def}}{=} \frac{1}{R_1(a)}(1 + d_1\rho + d_2\rho^2 + O(\rho^3)). \qquad (5)$$

Let us transform the expression for the coefficient d_2 at ρ^2 in decomposition (5). Introducing into this expression the Schwartzian $\{f,a\} = \frac{f'''(a)}{f'(a)} - \frac{3}{2}\left(\frac{f''(a)}{f'(a)}\right)^2$, we will have

$$d_2 = -b_2 + b_1^2 \stackrel{(3),(4)}{=} -\frac{1}{2}\operatorname{Re}\left[\left(\frac{f'''(a)}{f'(a)} - \frac{3}{2}\left(\frac{f''(a)}{f'(a)}\right)^2\right)e^{i2\theta}\right] - \frac{1}{4}\operatorname{Re}\left[\left(\frac{f''(a)}{f'(a)}\right)^2 e^{i2\theta}\right] - \\ - \frac{1}{2}\left[\operatorname{Re}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\right]^2 + \operatorname{Re}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\frac{2\operatorname{Re}(\overline{a}e^{i\theta})}{1-|a|^2} + \frac{1}{1-|a|^2} + \left[\operatorname{Re}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\right]^2 - \\ - 2\operatorname{Re}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\frac{2\operatorname{Re}(\overline{a}e^{i\theta})}{1-|a|^2} + 4\left[\frac{\operatorname{Re}(\overline{a}e^{i\theta})}{1-|a|^2}\right]^2 = -\frac{1}{2}\operatorname{Re}[\{f,a\}e^{i2\theta}] + \\ + \frac{1}{4}\left[\operatorname{Im}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\right]^2 + \frac{1}{4}\left[\operatorname{Re}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\right]^2 - \operatorname{Re}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\frac{2\operatorname{Re}(\overline{a}e^{i\theta})}{1-|a|^2} + \\ + 4\left[\frac{\operatorname{Re}(\overline{a}e^{i\theta})}{1-|a|^2}\right]^2 + \frac{1}{1-|a|^2} = -\frac{1}{2}\operatorname{Re}\left[\{f,a\}e^{i2\theta} - \frac{2}{1-|a|^2}\right] + \\ + \frac{1}{4}\left[\operatorname{Im}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\right]^2 + \left(\operatorname{Re}\left[\left(\frac{1}{2}\frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{1-|a|^2}\right)e^{i\theta}\right]\right)^2.$$

Hence we obtain the estimate for the coefficient d_2 under the assumption that $|\{f,a\}| \le 2(1-|a|^2)^{-1}$ $(\le 2(1-|a|^2)^{-2})$,

$$d_2 \ge -\frac{1}{2}\left[\operatorname{Re}(\{f,a\}e^{i2\theta}) - \frac{2}{1-|a|^2}\right] \ge 0.$$

The latter means that the surface with the equation $\omega = 1/R_1(\zeta)$ over the disk E will be convex downwards. By the same token we have proved the following

Theorem 1. *If a regular function $f(\zeta)$ satisfies in the disk E the condition*

$$|\{f(\zeta),\zeta\}| \le 2/(1-|\zeta|^2), \quad \zeta \in E, \tag{6}$$

with the Schwartzian (Schwartz derivative) $\{f(\zeta),\zeta\}$, then the surface $\omega = \lambda(f(E), f(\zeta))$, constructed over the disk E, is convex downwards.

Let us note that the conclusion about the constancy of the sign of d_2 has been made with a certain reserve which is determined by the non-negativity of the values

$$A_1(a) = \left[\operatorname{Im}\left(\frac{f''(a)}{f'(a)}e^{i\theta}\right)\right]^2/4 \quad \text{and} \quad A_2(a) = \left(\operatorname{Re}\left[\left(\frac{1}{2}\frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{1-|a|^2}\right)e^{i\theta}\right]\right)^2.$$

The assertion of Theorem 1 will be valid under a weaker (but more cumbersome) constraint upon the Schwartzian in the form

$$|\{f(\zeta),\zeta\}| \le 2/(1-|\zeta|^2) + 2A_1(\zeta) + 2A_2(\zeta).$$

The functions with condition (6) belong to the Nehari class of functions satisfying the inequality

$$|\{f(\zeta),\zeta\}| \le 2/(1-|\zeta|^2)^2, \quad \zeta \in E. \tag{7}$$

For this class in the following Item we will obtain a result which is analogous to Theorem 1, but slightly weaker.

2. Theorem 2. *The surface for the density of the hyperbolic metric $\lambda_1(\zeta) = 1/R_1(\zeta)$ with the equation*

$$\omega = \lambda_1\left(\frac{\zeta+a}{1+\overline{a}\zeta}\right) = [|f'(T(\zeta))|(1-|T(\zeta)|^2)]^{-1}, \quad T(\zeta) = \frac{\zeta+a}{1+\overline{a}\zeta},$$

over a neighborhood of the point $a \in E$ is convex downwards under condition (7).

Proof. We can take advantage of the decomposition

$$\begin{aligned} f\left(\frac{\zeta+a}{1+\overline{a}\zeta}\right) &= f(a) + f'(a)(1-|a|^2)\zeta + \frac{1}{2}[f''(a)(1-|a|^2)^2 - f'(a)2\overline{a}(1-|a|^2)]\zeta^2 + \\ &+ \frac{1}{6}[f'''(a)(1-|a|^2)^3 - 6f''(a)\overline{a}(1-|a|^2)^2 + 6f'(a)\overline{a}^2(1-|a|^2)]\zeta^3 + \cdots = \\ &= f(a) + f'(a)(1-|a|^2)[\zeta + a_2\zeta^2 + a_3\zeta^3 + \cdots], \end{aligned} \tag{8}$$

where

$$a_2 = \frac{1}{2}\left[\frac{f''(a)}{f'(a)}(1-|a|^2) - 2\overline{a}\right], \quad a_3 = \frac{1}{6}\frac{f'''(a)}{f'(a)}(1-|a|^2)^2 - \frac{f''(a)}{f'(a)}\overline{a}(1-|a|^2) + \overline{a}^2, \tag{9}$$

and take into account that $|f'(a)|(1-|a|^2) = R_1(a)$. Then one can express $R_1[T(\zeta)]$ and $\lambda_1[T(\zeta)]$ via a_2 and a_3. Namely,

$$\begin{aligned} R_1(T(\zeta)) &= \left|f'_\zeta\left(\frac{\zeta+a}{1+\overline{a}\zeta}\right)\right|\left|\frac{d}{d\zeta}\left(\frac{\zeta+a}{1+\overline{a}\zeta}\right)\right|^{-1}\frac{|1+\overline{a}\zeta|^2 - |\zeta+a|^2}{|1+\overline{a}\zeta|^2} = \\ &= \left|f'_\zeta\left(\frac{\zeta+a}{1+\overline{a}\zeta}\right)\right|\frac{1+|a|^2|\zeta|^2 - |\zeta|^2 - |a|^2}{1-|a|^2} = R_1(a)|1 + 2a_2\zeta + 3a_3\zeta^2 + \cdots|(1-|\zeta|^2). \end{aligned}$$

Assuming $\zeta = \rho e^{i\theta}$, after a simple calculation we will have

$$\begin{aligned} R_1(T(\rho e^{i\theta}))/R_1(a) &= [(1+2a_2\rho e^{i\theta} + 3a_3\rho^2 e^{i2\theta} + \cdots)(1+2\overline{a}_2\rho e^{-i\theta} + 3\overline{a}_3\rho^2 e^{-i2\theta} + \cdots)]^{1/2}(1-\rho^2) = \\ &= [1 + 4\operatorname{Re}(a_2 e^{i\theta})\rho + 6\operatorname{Re}(a_3 e^{i2\theta})\rho^2 + 4|a_2|^2\rho^2 + \cdots]^{1/2}(1-\rho^2) = 1 + c_1\rho + c_2\rho^2 + O(\rho^3), \end{aligned}$$

where $c_1 = 2\,\mathrm{Re}(a_2 e^{i\theta})$, $c_2 = 3\,\mathrm{Re}(a_3 e^{i2\theta}) + 2|a_2|^2 - 2[\mathrm{Re}(a_2 e^{i\theta})]^2 - 1$. Now it is not difficult to obtain a decomposition for $\lambda_1[T(\zeta)]$. Indeed, we have

$$\lambda_1[T(\rho e^{i\theta})]R_1(a) = \frac{R_1(a)}{R_1(T(\rho e^{i\theta}))} = [1+c_1\rho+c_2\rho^2+O(\rho^3)]^{-1} = 1-c_1\rho+(c_1^2-c_2)\rho^2+O(\rho^3); \quad (10)$$

besides,

$$\begin{aligned} c_1^2 - c_2 &= 4[\mathrm{Re}(a_2 e^{i\theta})]^2 - 3\,\mathrm{Re}(a_3 e^{i2\theta}) - 2|a_2|^2 + 2[\mathrm{Re}(a_2 e^{i\theta})]^2 + 1 = \\ &= -3\,\mathrm{Re}[(a_3 - a_2^2)e^{i2\theta}] - 3\,\mathrm{Re}(a_2 e^{i\theta})^2 + 6[\mathrm{Re}(a_2 e^{i\theta})]^2 - 2|a_2|^2 + 1 = \\ &= -3\,\mathrm{Re}[(a_3 - a_2^2)e^{i2\theta}] + |a_2|^2 + 1. \end{aligned}$$

In order to apply condition (7), we take into account that by virtue of (9) we have

$$\begin{aligned} a_3 - a_2^2 &= \frac{1}{6}\frac{f'''(a)}{f'(a)}(1-|a|^2)^2 - \frac{f''(a)}{f'(a)}\overline{a}(1-|a|^2) + \overline{a}^2 - \frac{1}{4}\left(\frac{f''(a)}{f'(a)}\right)^2(1-|a|^2)^2 + \\ &+ \overline{a}(1-|a|^2)\frac{f''(a)}{f'(a)} - \overline{a}^2 = \frac{1}{6}\left[\frac{f'''(a)}{f'(a)} - \frac{3}{2}\left(\frac{f''(a)}{f'(a)}\right)^2\right](1-|a|^2)^2 = \frac{1}{6}\{f(a), a\}(1-|a|^2)^2 \quad (11) \end{aligned}$$

with the Schwartzian $\{f(a), a\}$ in the final equality. Therefore from representation (10) under fulfillment of (7) we have

$$R_1(a)\frac{\partial^2 \lambda_1[T(\rho e^{i\theta})]}{\partial \rho^2}\bigg|_{\rho=0} = 2\left[-\frac{1}{2}\,\mathrm{Re}(\{f(a), a\}e^{i2\theta})(1-|a|^2)^2 + |a_2|^2 + 1\right] \geq 2|a_2|^2 \geq 0.$$

The resulting equality to zero is possible only for

$$a_2 = 0 \;\Leftrightarrow\; \frac{f''(a)}{f'(a)} = \frac{2\overline{a}}{1-|a|^2},$$

i. e., when the point a is a critical one. If in a punctured neighborhood of the critical point a the relation $f''(\zeta)/f'(\zeta) \neq 2\overline{\zeta}(1-|\zeta|^2)^{-1}$ is fulfilled, then the point a will be the point of minimum. Otherwise the point a is related to a continuum of minimums. In all of the above cases we have substantiated the local convexity downwards of the surface $\omega = \lambda_1\big(\frac{\zeta+a}{1+\overline{a}\zeta}\big)$. □

Remark. We can show that two points of minimum on the surface with the equation $\omega = \lambda_1[T(\zeta)]$ cannot exist. To this end by means of a linear-fractional function it is necessary to carry out the minimum point to the origin and then demonstrate that on the radii for $\theta = c$ and $0 < \rho < 1$ we will have $\mathrm{Re}(a_2 e^{i\theta}) \leq 0 \Leftrightarrow \frac{\partial \lambda_1(\rho e^{i\theta})}{\partial \rho} \geq 0$. The exclusive case is the presence of a continuum of minimums for the conformal radius, which is related to a rectangular strip.

3. Theorem 3. *Under condition* (7) *in the intersections of the surface whose equation is*

$$\omega = \lambda(\mathcal{D}, b + re^{i\theta}) = [R(f(E), f(a) + re^{i\theta})]^{-1}$$

and which is constructed over the domain $\mathcal{D}$, with the planes passing through the point b orthogonally to the domain $\mathcal{D}$, curves can be found which are convex downwards in a neighborhood of the point $(b, \lambda(\mathcal{D}, b))$.

Proof. Let us construct the decomposition of the conformal radius over a neighborhood of the point $b = f(a)$ in the domain $\mathcal{D} = f(E)$. Let us use representation (8), (9), i. e.,

$$\mathcal{F}^{-1}(\zeta) = f\left(\frac{\zeta + a}{1 + \overline{a}\zeta}\right) = f(a) + f'(a)(1-|a|^2)[\zeta + a_2\zeta^2 + a_3\zeta^3 + \cdots],$$

and obtain the decomposition of $R_2(b + re^{i\theta}) \stackrel{\mathrm{def}}{=} R(\mathcal{D}, b + re^{i\theta})$ in powers of r.

Let us write the representation which relates the neighborhood of the point b from the domain $\mathcal{D}$ and the neighborhood of zero from the disk E,

$$b + re^{i\theta} = \mathcal{F}^{-1}(\zeta_r) = b + f'(a)(1-|a|^2)(\zeta_r + a_2\zeta_r^2 + a_3\zeta_r^3 + \cdots);$$

$$\frac{re^{i\theta}}{f'(a)(1-|a|^2)} = \zeta_r + a_2\zeta_r^2 + a_3\zeta_r^3 + \cdots \Rightarrow \zeta_r = \frac{re^{i\theta}}{f'(a)(1-|a|^2)} - a_2\left(\frac{re^{i\theta}}{f'(a)(1-|a|^2)}\right)^2 + \cdots$$

Next, we represent the expression for the conformal radius at the point ζ_r which corresponds to the point $b + re^{i\theta} \in \mathcal{D}$, and take into account that $(\mathcal{F}^{-1})'(0) = f'(a)(1-|a|^2) = R(\mathcal{D}, b)e^{-i\beta}$. We thus will have

$$\begin{aligned} R_2(b+re^{i\theta}) &= |(\mathcal{F}^{-1})'(\zeta_r)|(1-|\zeta_r|^2) = R(\mathcal{D},b)[(1+2a_2\zeta_r+3a_3\zeta_r^2+\cdots)(1+2\overline{a}_2\overline{\zeta}_r + \\ &+ 3\overline{a}_3\overline{\zeta}_r^2+\cdots)]^{1/2}(1-|\zeta_r|^2) = R(\mathcal{D},b)\left[1 + 4\operatorname{Re}\left(a_2\frac{re^{i(\theta+\beta)}}{R(\mathcal{D},b)}\right) - 4\operatorname{Re}\left(a_2^2\frac{r^2e^{i2(\theta+\beta)}}{R^2(\mathcal{D},b)}\right) + \right. \\ &\left. + 4|a_2|^2\frac{r^2}{R^2(\mathcal{D},b)} + 6\operatorname{Re}\left(a_3\frac{r^2e^{i2(\theta+\beta)}}{R^2(\mathcal{D},b)}\right)\right]^{1/2}\left(1 - \frac{r^2}{R^2(\mathcal{D},b)}\right) + O(r^3) = \\ &= R(\mathcal{D},b)\left[1 + 2\operatorname{Re}\left(a_2\frac{re^{i(\theta+\beta)}}{R(\mathcal{D},b)}\right) - 2\operatorname{Re}\left(a_2^2\frac{r^2e^{i2(\theta+\beta)}}{R^2(\mathcal{D},b)}\right) + 2|a_2|^2\frac{r^2}{R^2(\mathcal{D},b)} + \right. \\ &\left. + 3\operatorname{Re}\left(a_3\frac{r^2e^{i2(\theta+\beta)}}{R^2(\mathcal{D},b)}\right) - 2\left[\operatorname{Re}\left(a_2\frac{re^{i(\theta+\beta)}}{R(\mathcal{D},b)}\right)\right]^2 - \frac{r^2}{R^2(\mathcal{D},b)} + O(r^3)\right]. \end{aligned}$$

Finally, we obtain the decomposition

$$R_2(b+re^{i\theta}) = R(\mathcal{D},b)\left(1 + c_1\frac{r}{R(\mathcal{D},b)} + c_2\frac{r^2}{R^2(\mathcal{D},b)} + O(r^3)\right), \tag{12}$$

where

$$\begin{aligned} c_1 &= 2\operatorname{Re}(a_2e^{i(\theta+\beta)}), \\ c_2 &= 3\operatorname{Re}[(a_3-a_2^2)e^{i2(\theta+\beta)}] - |a_2|^2 + 2|a_2|^2 - 1 = 3\operatorname{Re}[(a_3-a_2^2)e^{i2(\theta+\beta)}] + |a_2|^2 - 1. \end{aligned} \tag{13}$$

Decomposition (12) can be reached by means of the following representation in the domain $\mathcal{D} = f(E)$, $b = f(a)$,

$$\begin{aligned} R_2(z) = R_2(b+re^{i\theta}) = R_2(b) + 2\operatorname{Re}\left(\frac{\partial R_2}{\partial z}(b)e^{i\theta}\right)r + \\ + \left[\operatorname{Re}\left(\frac{\partial^2 R_2}{\partial z^2}(b)e^{i2\theta}\right) + \frac{\partial^2 R_2}{\partial z\partial\overline{z}}(b)\right]r^2 + O(r^3), \end{aligned} \tag{12$'$}$$

which is similar to (2$'$).

Now one can easily obtain the representation for $1/R_2(b+re^{i\theta})$. Indeed,

$$R(\mathcal{D},b)\lambda_2(b+re^{i\theta}) \overset{\text{def}}{=} R(\mathcal{D},b)/R_2(b+re^{i\theta}) = 1 - c_1\frac{r}{R(\mathcal{D},b)} + (c_1^2 - c_2)\frac{r^2}{R^2(\mathcal{D},b)} + O(r^3),$$

and $C_2 \overset{\text{def}}{=} c_1^2 - c_2 = -3\operatorname{Re}[(a_3-a_2^2)e^{i2(\theta+\beta)}] - |a_2|^2 + 1 + 4[\operatorname{Re}(a_2e^{i(\theta+\beta)})]^2$.

Under condition (7) the estimate for C_2 can be obtained in the form:

$$C_2 \geq 4[\operatorname{Re}(a_2e^{i(\theta+\beta)})]^2 - |a_2|^2,$$

because $-3\operatorname{Re}[(a_3-a_2^2)e^{i2(\theta+\beta)}] = -\frac{1}{2}\operatorname{Re}[\{f(a),a\}e^{i2(\theta+\beta)}](1-|a|^2)^2 \geq -1$. Hence we have $C_2 \geq 0$ for $\theta = \theta_0 \equiv -\beta - \arg a_2$ or for $\theta = \theta_0 + \pi$. Along these directions we have

$$(\lambda_2)''_{r^2}(b \pm re^{i\theta_0}) = \left.\frac{d^2\lambda(\mathcal{D}, b \pm re^{i\theta_0})}{dr^2}\right|_{r=0} \geq 0.$$

By the same token we have determined directions along which the line of cut of the surface is a convex curve in a neighborhood of every point $(b, \lambda(\mathcal{D}, b))$. $\square$

4. The form (13) for the coefficient c_2 from decomposition (12) enables us to state the following

Theorem 4 (see [2]). *The following equivalencies are valid*: (1.1) $\Leftrightarrow$ (1.2) $\Leftrightarrow$ (1.3).

As concerns [2], here the proof will be changed.

(1.1) $\Rightarrow$ (1.2). Since $f(\zeta)$ is a convex function, the function

$$\phi(\omega) = \left[f\left(\frac{\omega+\zeta}{1+\overline{\zeta}\omega}\right) - f(\zeta)\right] \Big/ [f'(\zeta)(1-|\zeta|^2)] = \omega + a_2\omega^2 + a_3\omega^3 + \cdots, \tag{14}$$

whose coefficients are defined by formulas (9) for $a = \zeta$, is also convex. Let us take into account that from (14) the next decomposition follows

$$\frac{\omega\phi''(\omega)}{\phi'(\omega)} + 1 = 1 + 2a_2\omega + (6a_3 - 4a_2^2)\omega^2 + \cdots \tag{15}$$

The function $\Omega(w)$ which transfers the right halfplane into itself with the correspondence of points $\pm 1 \mapsto \pm 1$, $\infty \mapsto \frac{1+e^{i\alpha}}{1-e^{i\alpha}} = i \operatorname{ctg} \frac{\alpha}{2}$ (i. e., $\frac{\Omega(\infty)-1}{\Omega(\infty)+1} = e^{i\alpha}$), has the form

$$\frac{\Omega-1}{\Omega+1} = e^{i\alpha}\frac{w-1}{w+1} \Rightarrow \Omega(w) = \frac{(1+e^{i\alpha})w + 1 - e^{i\alpha}}{(1-e^{i\alpha})w + 1 + e^{i\alpha}} = \left[1 + \frac{1+e^{i\alpha}}{2}(w-1)\right]\left[1 + \frac{1-e^{i\alpha}}{2}(w-1)\right]^{-1}.$$

One can easily write out the decomposition of the function $\Omega(w)$ in powers of $w-1$

$$\Omega(w) = 1 + e^{i\alpha}(w-1) - \frac{e^{i\alpha}(1-e^{i\alpha})}{2}(w-1)^2 + \cdots \tag{16}$$

Following [5], we construct the decomposition for the superposition of functions (16) and (15):

$$\Omega\left[\omega\frac{\phi''(\omega)}{\phi'(\omega)} + 1\right] = 1 + e^{i\alpha}\omega\frac{\phi''(\omega)}{\phi'(\omega)} - \frac{e^{i\alpha}(1-e^{i\alpha})}{2}\left(\omega\frac{\phi''(\omega)}{\phi'(\omega)}\right)^2 + \cdots = 1 + e^{i\alpha}2a_2\omega +$$
$$+ e^{i\alpha}(6a_3 + 4a_2^2)\omega^2 - \frac{e^{i\alpha}(1-e^{i\alpha})}{2}4a_2^2\omega^2 + \cdots = 1 + 2a_2e^{i\alpha}e^{i\varphi}\tau + A_2\tau^2 + \cdots,$$

where $\tau = e^{-i\varphi}\omega$, $A_2 = 2[3(a_3 - a_2^2) + a_2^2e^{i\alpha}]e^{i\alpha}e^{i2\varphi}$.

Since for the coefficients of the function with positive real part in the disk E we have the estimate $|A_2| \le 2$ (see [6], p. 199), from the expression for coefficient A_2 we obtain

$$|A_2|/2 = |3(a_3 - a_2^2)e^{i\gamma} + a_2^2e^{i(\alpha+\gamma)}| \le 1, \quad \text{where} \quad \gamma = 2\varphi + \alpha.$$

As for the free real parameters, we can dispose them so that

$$\gamma = -\arg(a_3 - a_2^2), \quad \alpha = -\arg a_2^2 - \gamma.$$

Then the previous inequality can be written in the form $3|a_3 - a_2^2| + |a_2|^2 \le 1$ and with regard for expressions (9), (11) passes into (1.2).

Let us construct the two extremal functions for which in (1.2) the equality takes place.

The first of these functions satisfies the equation

$$\zeta\frac{f''(\zeta)}{f'(\zeta)} + 1 = \frac{1+\zeta e^{i\alpha}}{1-\zeta e^{i\alpha}}, \quad \alpha = \text{const} \in \mathbb{R}, \ \Rightarrow \ f'(\zeta) = \frac{c_1}{(1-\zeta e^{i\alpha})^2},$$
$$f(\zeta) = \frac{c_1e^{-i\alpha}}{1-\zeta e^{i\alpha}} + c_2 \ \Rightarrow \ f_0(\zeta) \stackrel{\text{def}}{=} \frac{a\zeta + b}{1-\zeta e^{i\alpha}}.$$

In addition, $a_2(\zeta)$ for the function $f_0\big(\frac{\omega+\zeta}{1+\overline{\zeta}\omega}\big)$ has the form $\frac{1}{2}\big(\frac{2e^{i\alpha}}{1-\zeta e^{i\alpha}}(1-|\zeta|^2)-2\overline{\zeta}\big)$ and therefore

$$|a_2(\zeta)|^2 = \left|\frac{1-|\zeta|^2}{1-\zeta e^{i\alpha}} - \overline{\zeta}e^{-i\alpha}\right|^2 = \left|\frac{1-\overline{\zeta}e^{-i\alpha}}{1-\zeta e^{i\alpha}}\right| = 1,$$

i.e., inequality (1.2) turns into equality. In the left-hand side in (1.2) for the linear-fractional function $f_0\big(\frac{\omega+\zeta}{1+\overline{\zeta}\omega}\big)$ we will have zero.

The second of these functions satisfies the equation

$$\zeta\frac{f''(\zeta)}{f'(\zeta)} + 1 = \frac{1+\zeta^2 e^{i2\alpha}}{1-\zeta^2 e^{i2\alpha}} \;\Rightarrow\; f'(\zeta) = \frac{2c_1 e^{i\alpha}}{1-\zeta^2 e^{i2\alpha}} \;\Rightarrow\; f_1(\zeta) \stackrel{\text{def}}{=} c_1 \ln\frac{1+\zeta e^{i\alpha}}{1-\zeta e^{i\alpha}} + c_2.$$

For the function $f_1\big(\frac{\omega+\zeta}{1+\overline{\zeta}\omega}\big)$ we transform the expression

$$a_2(\zeta) = \frac{1}{2}\left[\frac{2\zeta e^{i2\alpha}}{1-\zeta^2 e^{i2\alpha}}(1-|\zeta|^2) - 2\overline{\zeta}\right] = e^{i\alpha}\frac{\zeta e^{i\alpha}-\overline{\zeta}e^{-i\alpha}}{1-\zeta^2 e^{i2\alpha}}.$$

On the other hand,

$$\left\{f_1\left(\frac{\omega+\zeta}{1+\overline{\zeta}\omega}\right), \frac{\omega+\zeta}{1+\overline{\zeta}\omega}\right\}\bigg|_{\omega=0} = \{f_1(\zeta),\zeta\} = \left(\frac{2\zeta e^{i2\alpha}}{1-\zeta^2 e^{i2\alpha}}\right)' - \frac{1}{2}\frac{4\zeta^2 e^{i4\alpha}}{(1-\zeta^2 e^{i2\alpha})^2} = \frac{2e^{i2\alpha}}{(1-\zeta^2 e^{i2\alpha})^2}.$$

Thus, we have

$$\frac{1}{2}(1-|\zeta|^2)^2|\{f_1(\zeta),\zeta\}| = \frac{(1-|\zeta|^2)^2}{|1-\zeta^2 e^{i2\alpha}|^2} = 1 - \frac{|\zeta e^{i\alpha}-\overline{\zeta}e^{-i\alpha}|^2}{|1-\zeta^2 e^{i2\alpha}|^2} = 1-|a_2(\zeta)|^2,$$

and in (1.2) equality is attained for all $\zeta\in E$.

(1.2) $\Rightarrow$ (1.1). Under fulfillment of (1.2) we obtain the following implications in the disk E:

$$\left|\frac{f''(\zeta)}{f'(\zeta)}(1-|\zeta|^2) - 2\overline{\zeta}\right|^2 \le 4 \;\Rightarrow\; \left|\frac{f''(\zeta)}{f'(\zeta)}\right|^2(1-|\zeta|^2)^2 - 4\operatorname{Re}\left(\zeta\frac{f''(\zeta)}{f'(\zeta)}\right)(1-|\zeta|^2) + 4|\zeta|^2 \le 4 \;\Rightarrow$$
$$\Rightarrow\; \operatorname{Re}\left(\zeta\frac{f''(\zeta)}{f'(\zeta)}+1\right) \ge \frac{1}{4}\left|\frac{f''(\zeta)}{f'(\zeta)}\right|^2(1-|\zeta|^2) \ge 0 \;\Rightarrow\; (1.1).$$

(1.2) $\Rightarrow$ (1.3). Let us take into account the form (13) for the coefficient c_2 from decomposition (12). By virtue of (11) and (9) we have

$$a_3 - a_2^2 = \frac{1}{6}\{f(a),a\}(1-|a|^2)^2, \quad a_2 = \frac{1}{2}\left[\frac{f''(a)}{f'(a)}(1-|a|^2) - 2\overline{a}\right].$$

Therefore from (12), (13), and (1.2) we obtain the inequality

$$\frac{\partial^2 R_2(b+re^{i\theta})}{\partial r^2}\bigg|_{r=0} = \frac{2c_2}{R(\mathcal{D},b)} \le (1-|a_2|^2+|a_2|^2-1)\frac{2}{R(\mathcal{D},b)} = 0,$$

which ensures the convexity upwards of the surface $\omega = R(\mathcal{D},b)$ over every point $b\in\mathcal{D}$.

(1.3) $\Rightarrow$ (1.2). The condition of the convexity upwards of the surface of conformal radius can be written with regard for decomposition (12′) as follows

$$\operatorname{Re}\left(\frac{\partial^2 R}{\partial z^2}e^{i2\theta}\right) \le -\frac{\partial^2 R}{\partial z\partial\overline{z}}, \quad 0\le\theta\le 2\pi, \;\Leftrightarrow\; \left|\frac{\partial^2 R}{\partial z^2}\right| \le -\frac{\partial^2 R}{\partial z\partial\overline{z}}.$$

Let us expose the results of the straightforward calculation for $R(f(E),z) = |f'(\zeta)|(1-|\zeta|^2)$ and $z = f(\zeta)$, $\zeta = f^{-1}(z)$. We will have

$$\frac{\partial R}{\partial z} = \frac{\partial R}{\partial \zeta}\frac{1}{f'(\zeta)} = \sqrt{\frac{\overline{f'(\zeta)}}{f'(\zeta)}}\Big[\frac{f''(\zeta)}{2f'(\zeta)}(1-|\zeta|^2) - \overline{\zeta}\Big], \quad R\Big|\frac{\partial^2 R}{\partial z^2}\Big| = \frac{1}{2}|\{f(\zeta),\zeta\}|(1-|\zeta|^2)^2,$$

$$R\frac{\partial^2 R}{\partial z\partial\overline{z}} = \frac{1}{4}\Big|\frac{f''(\zeta)}{f'(\zeta)}\Big|^2(1-|\zeta|^2)^2 - (1-|\zeta|^2)\Big[\mathrm{Re}\Big(\zeta\frac{f''(\zeta)}{f'(\zeta)}\Big)+1\Big] = \Big|\frac{f''(\zeta)}{2f'(\zeta)}(1-|\zeta|^2) - \overline{\zeta}\Big|^2 - 1.$$

Then

$$R\Big|\frac{\partial^2 R}{\partial z^2}\Big| \le -R\frac{\partial^2 R}{\partial z\partial\overline{z}} \;\Leftrightarrow\; \frac{1}{2}|\{f(\zeta),\zeta\}|(1-|\zeta|^2)^2 \le 1 - \Big|\frac{f''(\zeta)}{2f'(\zeta)}(1-|\zeta|^2) - \overline{\zeta}\Big|^2,$$

i. e., we have obtained inequality (1.2). □

Theorem 5. *If the surface of conformal radius $R(f(E), f(\zeta))$, constructed over the disk E, is convex upwards, then the domain $\mathcal{D} = f(E)$ is convex. The inverse assertion is not true in the general case.*

Proof. Let us calculate the derivatives which participate in the condition of convexity upwards of the surface with the equation $\omega = |f'(\zeta)|(1-|\zeta|^2)$. This condition follows from representation $(2')$ and has the form (similar to $b_2 \le 0$ from (4))

$$\mathrm{Re}\Big(\frac{\partial^2 R}{\partial\zeta^2}e^{i2\theta}\Big) + \frac{\partial^2 R}{\partial\zeta\partial\overline{\zeta}} \le 0, \quad 0 \le \theta \le 2\pi, \;\Leftrightarrow\; \Big|\frac{\partial^2 R}{\partial\zeta^2}\Big| \le -\frac{\partial^2 R}{\partial\zeta\partial\overline{\zeta}}.$$

We obtain consecutively

$$\frac{\partial R}{\partial\zeta} = |f'(\zeta)|\Big(\frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)}(1-|\zeta|^2) - \overline{\zeta}\Big),$$

$$\frac{\partial^2 R}{\partial\zeta^2} = |f'(\zeta)|\Big[\frac{1}{2}\Big(\Big(\frac{f''(\zeta)}{f'(\zeta)}\Big)' + \frac{1}{2}\Big(\frac{f''(\zeta)}{f'(\zeta)}\Big)^2\Big)(1-|\zeta|^2) - \overline{\zeta}\frac{f''(\zeta)}{f'(\zeta)}\Big],$$

$$\frac{\partial^2 R}{\partial\zeta\partial\overline{\zeta}} = |f'(\zeta)|\Big[\frac{1}{4}\Big|\frac{f''(\zeta)}{f'(\zeta)}\Big|^2(1-|\zeta|^2) - \frac{1}{2}\Big(\overline{\frac{f''(\zeta)}{f'(\zeta)}}\Big)\overline{\zeta} - \frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)}\zeta - 1\Big] =$$

$$= \frac{|f'(\zeta)|}{1-|\zeta|^2}\Big(\Big|\frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)}(1-|\zeta|^2) - \overline{\zeta}\Big|^2 - 1\Big).$$

Therefore the inequality

$$\frac{1-|\zeta|^2}{|f'(\zeta)|}\Big|\frac{\partial^2 R}{\partial\zeta^2}\Big| \le -\frac{1-|\zeta|^2}{|f'(\zeta)|}\frac{\partial^2 R}{\partial\zeta\partial\overline{\zeta}}$$

can be rewritten in the equivalent form

$$(1-|\zeta|^2)\Big|\frac{1}{2}\Big[\Big(\frac{f''(\zeta)}{f'(\zeta)}\Big)' + \frac{1}{2}\Big(\frac{f''(\zeta)}{f'(\zeta)}\Big)^2\Big](1-|\zeta|^2) - \overline{\zeta}\frac{f''(\zeta)}{f'(\zeta)}\Big| \le 1 - \Big|\frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)}(1-|\zeta|^2) - \overline{\zeta}\Big|^2.$$

Under the fulfillment of this inequality we will have (see the transition in (1.2) ⇒ (1.1))

$$\Big|\frac{1}{2}\frac{f''(\zeta)}{f'(\zeta)}(1-|\zeta|^2) - \overline{\zeta}\Big|^2 \le 1 \;\Leftrightarrow\; \mathrm{Re}\Big(\zeta\frac{f''(\zeta)}{f'(\zeta)}\Big) \ge -1,$$

i. e., the convexity of the function $f(\zeta)$ and the convexity of the domain $f(E) = \mathcal{D}$ are ensured.

In the general case, the inverse transition from the convexity of the domain $\mathcal{D} = f(E)$ to the convexity upwards of the surface with the equation $\omega = R(f(E), f(\zeta))$ over the disk E cannot be realized. It suffices to verify this fact by the example of the domain $f_r(E)$ with $f_r(\zeta) = \frac{1+r\zeta}{1-r\zeta}$, $0 < r \le 1$. The domain $f_r(E)$ will be a disk from the right halfplane which is the limit position of this disk as $r \to 1$. On the limit surface of conformal radius, ∞ will lie over the point $\zeta = 1$.

The line of cut of this surface along the real diameter is the graph of a function which is convex downwards, because

$$|f_1'(\xi)|(1-\xi^2) = \frac{2(1-\xi^2)}{(1-\xi)^2} = \frac{2(1+\xi)}{1-\xi} = \omega, \quad \omega' = \frac{4}{(1-\xi)^2},$$

and therefore $\omega'' = 8/(1-\xi)^2 > 0$, $\xi \in [-1,1)$. For r close to the unit we have

$$\omega = |f_r'(\xi)|(1-\xi^2) = \frac{2r(1-\xi^2)}{(1-r\xi)^2}, \quad \omega' = \frac{4r(r-\xi)}{(1-r\xi)^3}, \quad \omega'' = \frac{4r(3r^2-1-2r\xi)}{(1-r\xi)^4},$$

and the second derivative will change its sign from (+) for $\xi \in [-1, (3r^2-1)(2r)^{-1})$ to (−) for $\xi \in ((3r^2-1)(2r)^{-1}, 1]$, i. e., no convexity is present. The finite maximum of the conformal radius, which is equal to $2r(1-r^2)^{-1}$, is situated over the point $\xi = r$. □

5. In the conclusion, we give a geometrical interpretation of the inequality

$$\left|\frac{f''(\zeta)}{f'(\zeta)}\right| < \frac{2|\zeta|}{1-|\zeta|^2}, \quad \zeta \in E, \tag{17}$$

which implies the uniqueness of the root $\zeta = 0$ of the equation

$$\frac{f''(\zeta)}{f'(\zeta)} = \frac{2\overline{\zeta}}{1-|\zeta|^2}$$

under the assumption $f''(0) = 0$. Inequality (17) is frequently used in the substantiation of the theorems about uniqueness of the solution of the external boundary value problem (see [7]–[9]).

To interpret (17) we recall expression (3) for the coefficient b_1 in decomposition (2) and take into account that

$$\frac{1}{R_1(a)} \frac{\partial R(f(E), f(a+\rho e^{i\theta}))}{\partial \rho}\bigg|_{\rho=0} = b_1 = \operatorname{Re}\left[\left(\frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{1-|a|^2}\right)e^{i\theta}\right].$$

Put $\theta = \arg a$. Then we can write

$$b_1 = \operatorname{Re}\left(\frac{f''(a)}{f'(a)} e^{i\arg a}\right) - \frac{2|a|}{1-|a|^2};$$

whence by virtue of (17) we obtain $b_1 < 0$. The latter means that the line which is the intersection of the surface of conformal radius with the surface passing perpendicularly to the disk along any ray, will be the graph of a strictly decreasing function. All these lines descend from the point of maximum over $\zeta = 0$ to a point of the boundary circle. In other words, ray cuts of the surface of conformal radius form monotone lines. The curvature sign may change on these lines. On such a surface some hollows may be met.

A more complex family of lines is related to Theorem 3. In this case, equations of lines with a positive coefficient at r arise, but on the surface for the density of the hyperbolic metric (i. e., for inverted conformal radius) over the domain $f(E)$. The existence of such lines leads to the uniqueness of the critical point of the function $\lambda(f(E), z) = 1/R(f(E), z)$ under condition (7). Then the critical point of the conformal radius $R(f(E), z)$ will be unique: Equations for the critical points of the surfaces $\omega = R(\mathcal{D}, f(\zeta))$ and $\omega = 1/R(\mathcal{D}, f(\zeta))$ differ by the sign in the left side ($b_1 = 0$ from (2), (3) $\Leftrightarrow -b_1 = 0$ from (5)).

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