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VARIATIONAL METHOD AND OPTIMAL CONTROL IN SOLVING THE MOCANU PROBLEM

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In the present article the problem on estimation of the module of ratio between a univalent function and its derivative, both given in two different real points of the circle $E = \{z : |z| < 1\}$, is solved. In solving the problem, we used variational method by G.M. Goluzin and a combination of the parametric Loewner representations method and Pontryagin's maximum principle.

We denote by S a class of all holomorphic functions $f(z) = z + a_2 z^2 + \cdots$ univalent in the circle E.

In 1966, at the Conference on Analytic Functions in Lodz (Poland), P. Mocanu posed an extremal problem which can be reduced to the problem on the maximum

$$I(f) = \left|\frac{f'(z_1)}{f(z_2)}\right|, \quad f \in S.$$

In 1974 P. Mocanu, M. Reade and E. Zlotkiewicz (see [1]) estimated $I(f)$ for typically-real f and real z_1, z_2. Later on there results were generalized by D. Ripeanu for complex z_1, z_2 from special domains in E. Since the extremal functions in [1] were univalent, by same token they simultaneously solved the problem on the maximum of $I(f)$ for real z_1, z_2 in the class $S_R \subset S$ of univalent functions with real coefficients. For a more general problem on the extremum of $f'(r_1)$ for fixed values of $f(r_1)$, $f(r_2)$ in [2] exact majorants were constructed by means of the method of modules of families of curves.

Since the appearance of the article [3], in the theory of univalent functions the methods of optimal control found successful application. Deep results in this direction were obtained by D.V. Prokhorov, who suggested to consider the sets of values of systems of functionals in the capacity of domains of attainability (reachable sets) for controllable systems induced by the Loewner–Kufarev's equation. For most wide reference on these results the reader can use the paper [4] and the monograph [5]. However, the problems on estimation of functionals, which are reducible to construction of projections of attainability sets to various hypersurfaces, lead to boundary value problems for controllable systems. Application of the variational method along with the optimal control method enables to avoid difficulties. This combination was suggested in [6] and allowed to reduce the boundary value problems to the Cauchy problems. In the present article, by means of the variational method and the method of optimal control, we solve the problem on estimation of $I(f)$ in the class S for real z_1, z_2.

1. Differential equations for extremal functions

Let $f(z)$ be a function supplying the maximum to the functional $I(f)$. By means of Goluzin's variational formula (see [7], p. 120), we establish that $f(z) \in S$ maps the circle E onto the plane