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A CRITERION FOR SPATIAL PROJECTIVITY OF OPERATOR ALGEBRAS POSSESSING A CANONICAL REPRESENTATION

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1. Introduction

Let $A \subset \mathcal{B}(H)$ be an arbitrary (in general, nonselfadjoint) operator algebra closed with respect to the topology of uniform convergence. Then the Hilbert space H is a left A-module with the intrinsic multiplication $a \cdot h = a(h)$, $a \in A$, $h \in H$. We investigate the problem on conditions under which A is spatially projective, i. e., the module H is projective.

Recall that a left A-module H is said to be *projective* if any surjective morphism of an arbitrary Banach A-module onto H, which possesses a bounded right-inverse operator, has also a right-inverse morphism of A-modules (see [1], p. 361). In this article, we denote by $\dot{\otimes}$ and $\oplus$ the Hilbert tensor product and the orthogonal l^2-sum of Hilbert spaces, respectively. We denote by $\widehat{\otimes}$ the projective tensor product of Banach spaces. It turns out that the module H is projective if and only if the *canonical projection* $\pi : A_+ \widehat{\otimes} H \to H : a \widehat{\otimes} h \mapsto a \cdot h$ has a right-inverse morphism of modules (here the space $A_+ \widehat{\otimes} H$ is considered as a left A-module with multiplication defined by $a \cdot (b \widehat{\otimes} h) = ab \widehat{\otimes} h$, $a \in A$, $b \in A_+$, $h \in H$). In the case of a unital algebra A, one can consider also the canonical projection $\pi : A \widehat{\otimes} H \to H$.

The basic results concerning the problem of spatial projectivity of selfadjoint operator algebras were obtained in [2], [3]. Let the algebra $A \subset \mathcal{B}(H)$ be selfadjoint. Recall that a left module H over C*-algebra A is called *a Hilbert module* if it is a Hilbert space and, for any $x, y \in H$, $a \in A$, the following condition holds: $\langle a \cdot x, y \rangle = \langle x, a^* \cdot y \rangle$. Projectors $p, q \in A$ are said to be *Murray–von Neumann equivalent* if an element $u \in A$ exists such that $p = u^*u$ and $q = uu^*$. A projector $q \in A$ is said to be *elementary* if the subalgebra qAq is one-dimensional. In that case one can consider the left A-module $H_q = Aq$, which is the Hilbert module with respect to the scalar product $\langle aq, bq \rangle = qb^*aq$ (cf. [4], Chap. IV, 10). For any Hilbert space K_q we introduce the left Hilbert module $H_q \dot{\otimes} K_q$, the action on this module is correctly defined by the formula $a \cdot (bq \dot{\otimes} k) = abq \dot{\otimes} k$, $a \in A$, $bq \in Aq$, $k \in K_q$. Thus, an element $a \in A$ acts on $H_q \dot{\otimes} K_q$ as an operator of the form $\mathbf{a} \otimes \mathbf{1}$, where $\mathbf{a} \in \mathcal{B}(H_q)$.

As is known (see [3]), for any C*-algebra $A \subset \mathcal{B}(H)$, the Hilbert module H can be represented up to an isometric isomorphism between Hilbert modules in the form of the direct sum $(\oplus_{q \in \Lambda} H_q \dot{\otimes} K_q) \oplus H_0$, where Λ is the maximal set of pairwise Murray–von Neumann nonequivalent elementary projectors in A, H_q a system of left A-modules of the form Aq, K_q a system of arbitrary Hilbert spaces, and the Hilbert module H_0 contains no submodules isometrically isomorphic to a left submodule of the form H_q for any $q \in \Lambda$.

In addition, for any projector $q \in \Lambda$ and vectors $h_1, h_2 \in H_q$, the operator $h_1 \diamond h_2 \in A$ exists vanishing on the orthogonal complement to $H_q \dot{\otimes} K_q$ and sending an element $h \dot{\otimes} k \in H_q \dot{\otimes} K_q$ into $\langle h, h_2 \rangle h_1 \dot{\otimes} k$. A set of operators of the form $h_1 \diamond h_2$, $h_1 \in H_q$ for fixed $h_2 \in H_q$ will be called a