Bañados – Silk – West effect: geometrical explanation

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Point of collision. Basis

Standard choice: 1 timelike, 3 spaclike

Now: 2 lightlike, 2 spacelike a, b

lightlike vectors l^μ N^μ $l^\mu l_\mu = 0 = N^\mu N_\mu$

Normalization: $l^\mu N_\mu = -1$ $l^\mu a_\mu = 0 = N^\mu a_\mu$

$$t^\mu = \frac{l^\mu + N^\mu}{\sqrt{2}} \qquad t^\mu t_\mu = -1$$

$$e^\mu = \frac{l^\mu - N^\mu}{\sqrt{2}} \qquad e^\mu e_\mu = +1$$

Standard presentation

$$g_{\alpha\beta} = \eta_{ab} h_\alpha{}^{(a)} h_\beta{}^{(b)} \qquad \eta_{ab} = diag(-1,1,1,1)$$

$$h^{(0)\mu} = t^\mu \qquad h^{(1)\mu} = e^\mu \qquad h^{(2)\mu} = a^\mu \qquad h^{(3)\mu} = b^\mu$$

With the help of lightlike basis

$$g_{\alpha\beta} = -l_\alpha N_\beta - l_\beta N_\alpha + \sigma_{\alpha\beta}$$

$$\sigma_{\alpha\beta} = a_\alpha b_\beta + a_\beta b_\alpha$$

$$l^\alpha \sigma_{\alpha\beta} = N^\alpha \sigma_{\alpha\beta} = 0$$

Decomposition of the four-velocity of particle i

$$u_i^\mu = \frac{l^\mu}{2\alpha_i} + \beta_i N^\mu + s_i^\mu,\ s_i^\mu = A_i a^\mu + B_i b^\mu$$

$$\beta_i = -(u_i l), \qquad \alpha_i = -\frac{1}{2}(u_i N)^{-1}. \qquad (ab) = a^\mu b_\mu$$

All vectors future-directed $\alpha_i > 0$ $\beta_i > 0$

$$u^\mu u_\mu = -1 \qquad s_i^\mu s_{i\mu} = \frac{\beta_i}{\alpha_i} - 1.$$

Case $\beta_i = \alpha_i$ $s_i^\mu = 0$ radial motion (see below)

$$-(u_1 u_2) = \frac{1}{2}\left(\frac{\beta_1}{\alpha_2} + \frac{\beta_2}{\alpha_1}\right) - (s_1 s_2).$$

The energy in the centre of mass frame

$$E^2_{c.m} = -P^2 \quad P^\mu = m_1 u_1^\mu + m_2 u_2^\mu$$

$$E^2_{c.m.} = m_1^2 + m_2^2 - 2m_1 m_2 (u_1 u_2)$$

$$E^2_{c.m.} = m_1^2 + m_2^2 + m_1 m_2 \left[\frac{\beta_1}{\alpha_2} + \frac{\beta_2}{\alpha_1} - 2(s_1 s_2)\right].$$

Ingoing versus outgoing particles in the vicinity of the horizon: general approach

Case 1

Particle 1 moves in outward direction near horizon. Vector l^{μ} Is close

to horizon and becomes its generator when horizon is approached. Particle Does not cross horizon, component along N^{μ} almost vanishes

$$u_1^{\mu} = \frac{l^{\mu}}{2\alpha_1} + \beta_1 N^{\mu} + s_1^{\mu}, s_1^{\mu} = A_1 a^{\mu} + B_1 b^{\mu}$$

$$\beta_1 \to 0 \qquad \alpha_1 = \frac{\beta_1}{(ss)+1} \to 0.$$

$$E_{c.m.}^2 = m_1^2 + m_2^2 + m_1 m_2 [\frac{\beta_1}{\alpha_2} + \frac{\beta_2}{\alpha_1} - 2(s_1 s_2)] \to \infty$$

for any particle 2 Direct consequence of blue shift

Case 2

Both particles move towards the horizon

Frame of the centre of mass falls down with both particles

No direct consequence of blue shift

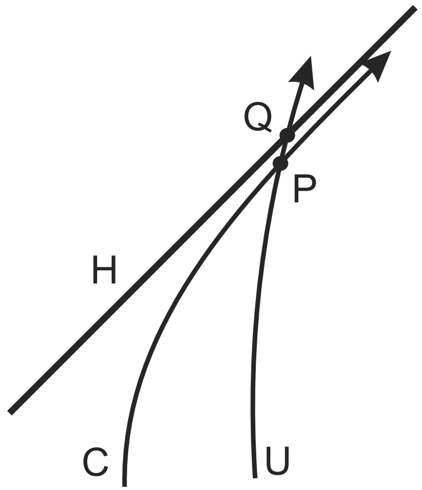
For any nonzero α_1 α_2

$\alpha_1 = 0$ Is NOT automatically, this is condition on parameters

If this condition is satisfied, $E^2_{c.m.} \to \infty.$ BSW effect

Relationship between energy and angular momentum or energy and Electric charge

Geometrically



C is critical particle

U is usual particle

Proper time grows unbound (T. Jacobson, Grib and Pavlov, O. Z.)

Kinematic censorship

Difference between cases 1 and 2

In terms of Kruskal coordinates

$$ds^2 = -CdUdV + \gamma_{ab}dx^a dx^b$$

C is some function bounded and analytical near horizon

U and V are lightlike coordinates

Future horizon U=0 $\quad u^U \sim \beta \to 0$

Normalizaiton condition $\quad (uu) = Cu^U u^V = -1$

$$\alpha \sim \beta \sim U \qquad \frac{dU}{d\tau} \sim U,$$

$$\tau \sim -\ln U \to \infty$$

For critical trajectory proper time diverges

Examples

Radial motion in Reissner-Nordstom black hole

$$ds^2 = -dt^2N^2 + \frac{dr^2}{N^2} + r^2 d\omega^2. \qquad d\omega^2 = \sin^2\theta d\phi^2 + d\theta^2$$

$$N^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

M is the black hole mass, Q is its charge

Pure radial motion

$$u^0 = \frac{X}{N^2 m}, u^1 = \varepsilon \frac{Z}{mN} \qquad X = E - \frac{qQ}{r}, Z = \sqrt{X^2 - m^2N^2},$$

E is energy

$\varepsilon = -1$ for direction towards horizon

$\varepsilon = +1$ for direction away from horizon

Killing vector $\xi^{\mu} = (1, 0, 0, 0)$

Two lightlike vectors

$$l^{\mu} = (1, N, 0, 0), N^{\mu} = \frac{1}{2}(\frac{1}{N^2}, -\frac{1}{N}, 0, 0),$$

$$(Nl) = -1 \qquad a_{\theta} = r \qquad b_{\phi} = r \sin\theta$$

$$-(\xi u) = \frac{X}{m} \qquad \beta = -(ul) = \frac{X - \varepsilon Z}{m} > 0.$$

$-(uN)N^2 = \frac{1}{2}\frac{X+\varepsilon Z}{m} > 0$ is finite for both signs of ε

$$\alpha = \frac{mN^2}{X + \varepsilon Z}. \qquad X^2 - Z^2 = m^2 N^2$$

$\beta = \alpha$ for radial motion

Case 1

$\varepsilon = +1$ $\qquad \alpha = \dfrac{X - Z}{m}.$

$\alpha > 0$ outside horizon

In the horizon limit $N \to 0$ $\qquad Z \to X$

For any parameters, $\alpha \to 0$

Case 2

$\varepsilon = -1$ On horizon, $Z_H = X_H$

$\alpha_H = \frac{2X_H}{m} \geq 0$ $-(\xi u)_H = \frac{X_H}{m} \geq 0$

If for particle 1 $X_H = 0$ $qQ = Er_H$

it follows that $\alpha_1 = \beta_1 \to 0$ when horizon is approached

Critical particle

$$E^2_{c.m.} \to \infty$$

More general setting

Consider vector timelike outside horizon ξ^μ $N^2 = -(\xi\xi) > 0$

$\xi^\mu = \frac{1}{2} l^\mu + N^2 N^\mu.$ Two properties hold.

1) Let, in near-horizon limit, condition $\beta_1 \to 0$ be satisfied (ξu) finite

Then, vector ξ^μ becomes lightlike in this limit

Proof. It follows from $u_i^\mu = \frac{l^\mu}{2\alpha_i} + \beta_i N^\mu + s_i^\mu, \; s_i^\mu = A_i a^\mu + B_i b^\mu$

$\beta \to 0$ and $\alpha \to 0$ becomes lightlike in this limit.

$(\xi u) \approx N^2 (Nu) = -\frac{N^2}{2\alpha}$ finite, so $N \to 0$

2) Let us, instead of $\beta \to 0$ assume that $(\xi u) \to 0$

Then, $\beta \to 0$ and ξ^{μ} becomes lightlike in this limit

Proof. Multiplying $\xi^{\mu} = \frac{1}{2} l^{\mu} + N^2 N^{\mu}$. by u_{μ} we obtain

$$\xi^{\mu} u_{\mu} = -\frac{\beta}{2} - \frac{N^2}{2\alpha}.$$

Both terms have the same sign (negative). Therefore, each of them vanishes separately in this limit, so

$\alpha \to 0$ $N^2 \to 0$

As a consequence $E^2_{c.m.} \to \infty$

ressembles situations with Killing vector but we did not use Killing equations

Axially-symmetric rotating black hole

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + dl^2 + g_{zz} dz^2$$

includes the Kerr and Kerr-Newman black holes

$$\dot{t} = u^0 = \frac{X}{N^2}, \; X = E - \omega L \qquad m = 1$$

$$\dot{\phi} = \frac{L}{g_{\phi\phi}} + \frac{\omega X}{N^2}, \qquad \dot{l} = \varepsilon \frac{Z}{N}, \; Z^2 = X^2 - N^2 \left(1 + \frac{L^2}{g_{\phi\phi}}\right)$$

$u_0 = -E$ energy $\qquad u_\phi = L$ angular

forward in time condition $\qquad \dot{t} > 0 \qquad E - \omega L > 0$

$$l_\mu = (-N^2, N, 0, 0) \qquad N_\mu = \frac{1}{2N^2}(-N^2, -N, 0, 0)$$

$$(Nl) = -1. \qquad \xi^\mu = \xi_1^\mu + \omega \xi_2^\mu$$

$\xi_1^\mu = (1,0,0,0)$ translations in time

$\xi_2^\mu = (0,0,1,0)$ rotations

On horizon $N = 0$ vector ξ^μ becomes lightlike

$$\xi^\mu \xi_\mu \to 0$$

Nonzero components of other vectors

$$b_z = \sqrt{g_{zz}} \qquad a_\phi = \sqrt{g_{\phi\phi}} \qquad a_0 = -\omega a_\phi$$

$$-(u\xi) = X \qquad \beta = -(ul) = \frac{X - \varepsilon Z}{m} > 0. \qquad \alpha = \frac{mN^2}{X + \varepsilon Z}.$$

$$\frac{\beta}{\alpha} = 1 + \frac{L^2}{g_{\phi\phi}}.$$

$\beta \neq \alpha$ but they proportional to each other:

If $\beta \to 0$ also $\alpha \to 0$

Case 1

$$\varepsilon = 1 \qquad \beta = \frac{X - Z}{m} \qquad \alpha = \frac{m N^2}{X + Z}.$$

$Z \to X$ again $\beta \to 0$ $\alpha \to 0$

for any particle

in horizon limit

$N \to 0$

Case 2

$$\beta = \frac{X + Z}{m} \qquad \alpha = \frac{m N^2}{X - Z}.$$

In hoizon limit $\beta \to \frac{2X_H}{m}$ Critical value $X_H = 0$

As $X = E - \omega L$ critical condition gives us

$$E_H = \omega_H L$$

Then, according to previous formulas,

$$E_{c.m.}^2 \rightarrow \infty$$

THANK YOU