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Group-Theoretic Matching of the Length and the Equality Principles in Geometry [1)]

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Abstract—The paper deals with the canonical deformed group of diffeomorphisms with a given length scale which describes the motion of single scales in a Riemannian space. This allows to measure the lengths of arbitrary curves implementing the length principle which was laid by B. Riemann at the foundation of geometry. We present a method of univocal extension of this group to a group which contains gauge rotations of vectors (the group of parallel translations) whose transformations leave unchanged the lengths of vectors and the corners between vectors. Thereby Klein's Erlangen Program—the principle of equality—is implemented for Riemannian spaces.

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INTRODUCTION

The Lobachevskii geometry and Gauss' papers on the theory of surfaces stimulated the intensive development of geometric theories in the 19th century. As a result, the papers of B. Riemann [1] and F. Klein [2] were published. These papers laid different principles at the foundation of the geometry: the *length principle* which requires the possibility to measure the lengths of arbitrary lines no matter how they are situated, and the *equality principle* which is established by coincidence of figures in the space by means of transformations belonging to a group of transformations of the space—the principal group of the geometry under consideration (according to F. Klein). According to E. Cartan ([3], P. 488), there is an *antagonism* between these two principles owing to the absence of any homogeneity in an arbitrary curved Riemannian space.

Earlier, attempts were made to overcome this antagonism by means of refusing from group structure of used transformations. E. Cartan suggested to consider a curved space as a *nonholonomic space* with the same principal (fundamental according to the terminology used by Cartan) group as the corresponding flat space ([3], P. 491). R. Sulanke and P. Wintgen [4] applied the *category theory* for description of curved spaces, and L. V. Sabinin [5] used *quasigroups* relying on the thesis that nonassociativity is an algebraic equivalent of the geometric notion of curvature.

It has been clarified later that so-called deformed generalized gauge groups introduced from physical arguments [6] can describe, and in two different ways, geometric structures of variable curvature [7], in particular, affinely connected spaces [8] and Riemannian spaces [9].

In the first case, the translation generators of a group coincide with covariant derivatives in a curved affinely connected space. The curvature and the torsion of the space are determined by the structure functions of the group, which are antisymmetric parts of the coefficients of the second order terms in the expansion of the multiplication law of the group with respect to parameters [8].

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