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COINCIDENCE OF TWO LINEARLY INVARIANT FAMILIES OF FUNCTIONS

V.V. Starkov and Z.Yu. Yakubovskiĭ

Introduction. The basic objects of investigation in this article are linearly invariant families (l. i. f.) of functions analytic in the disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The term *linear invariance of the family* $\mathfrak{M}$ of analytic and locally univalent in the disk Δ functions of the form $f(z) = z + a_2(f)z^2 + \cdots$ was introduced in [1]. It means that, along with every function $f \in \mathfrak{M}$, to this family the function

$$\Lambda_\phi[f](z) = \frac{f(\phi(z)) - f(\phi(0))}{f'(\phi(0))\phi'(0)} = z + \cdots \tag{0.1}$$

also belongs for any conformal automorphism ϕ of the disk Δ. Examples of l. i. f. are, in particular, the classes $V_{2\alpha}$, $\alpha \geq 1$, of functions with restricted boundary rotation (see [2], [3]), i. e., the classes of locally univalent functions for which the complete variation of the inclination angle of the tangent to the image of any circle $\{z = re^{it} : t \in [0, 2\pi)\}$, $r \in (0, 1)$, does not exceed $2\pi\alpha$. As is known, $V_2 = \mathcal{K}$; for $\alpha > 2$, the classes $V_{2\alpha}$ already contain non-univalent functions.

In the classes $V_{2\alpha}$ the following integral representation in known (see [2], [3]): $f \in V_{2\alpha}$ if and only if

$$f'(z) = \exp\Big[-2\int_0^{2\pi} \log(1 - ze^{it})d\mu(t)\Big], \quad z \in \Delta, \quad \log 1 = 0, \quad f(0) = 0, \tag{0.2}$$

where μ is a real function of bounded variation on $[0, 2\pi)$, which satisfies the conditions

$$\int_0^{2\pi} d\mu(t) = 1, \quad \int_0^{2\pi} |d\mu(t)| \leq \alpha. \tag{0.3}$$

The class of such functions μ is denoted by M_α.

By *order* of l. i. f. $\mathfrak{M}$ in [1] the number is called $\operatorname{ord}\mathfrak{M} = \sup_{f\in\mathfrak{M}} |a_2(f)|$, while by a *universal l. i. f.* of order α — the union $\mathcal{U}_\alpha$ of all l. i. f. $\mathfrak{M}$ for which $\operatorname{ord}\mathfrak{M} \leq \alpha$. As is known, $\mathcal{U}_\alpha = \emptyset$ for $\alpha < 1$, $\mathcal{U}_1 = \mathcal{K}$, $\operatorname{ord}\mathcal{K} = 1$, $\operatorname{ord} S = 2$, $\operatorname{ord} V_{2\alpha} = \alpha$.

Every function $f \in \mathcal{U}_\alpha$ can be represented as the uniform inside Δ limit of functions f_n possessing integral representation (0.2) with the real functions μ_n (see [4], § 6). However, the complete variation of all such functions μ_n already cannot be bounded on $[0, 2\pi)$ by a certain constant $\mathcal{M} = \mathcal{M}(f)$. Therefore it seems to be of interest to study l. i. f. consisting of the functions $f(z) = \lim_{n\to\infty} f_n(z)$ for which f_n have integral representations (0.2), while the complete variations of the corresponding functions μ_n are not bounded in total.

In the first part of this article we introduce and study l. i. f. of that kind $\mathcal{U}_\alpha(R)$, while in the second part we establish a connection between $\mathcal{U}_\alpha(R)$ and $V_{2\alpha}$.

1. *Linearly invariant family* $\mathcal{U}_\alpha(R)$.

Lemma 1. *For an integer* $n \ge 2$*, assume that*

$$d_1, \dots, d_n \in \mathbb{R}, \quad d_1 + \dots + d_n = 1, \tag{1.1}$$

$d_{n+j} = d_j$, $j \in \mathbb{N}$. *Then, for* $\alpha \ge 1$*, the following conditions are equivalent*

$$\frac{1-\alpha}{2} \le d_l + \dots + d_{l+p} \le \frac{1+\alpha}{2} \quad \forall (p+1), \; l \in \{1, \dots, n\} \tag{1.2}$$

and

$$|d_1 e^{i\gamma_1} + \dots + d_n e^{i\gamma_n}| \le \alpha \quad \forall (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n, \quad 0 \le \gamma_1 \le \dots \le \gamma_n \le 2\pi. \tag{1.3}$$

Proof. a) (1.2) $\Longrightarrow$ (1.3). Let $n = 2$. If $d_1, d_2 > 0$, then from (1.1) we obtain $|d_1 e^{i\gamma_1} + d_2 e^{i\gamma_2}| \le \alpha$. If $0 \le d_1, d_2 \le 0$, then $|d_1 e^{i\gamma_1} + d_2 e^{i\gamma_2}| \le d_1 + (-d_2) \le \frac{1+\alpha}{2} - \frac{1-\alpha}{2} = \alpha$.

For $n > 2$, we take the set $K = \{(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n : 0 \le \gamma_1 \le \dots \le \gamma_n \le 2\pi\}$, which is compact in $\mathbb{R}^n$. Therefore, for a fixed set of numbers $d_1, \dots, d_n$ a point $(\gamma_1^0, \dots, \gamma_n^0) \in K$ exists such that

$$\max_K |d_1 e^{i\gamma_1} + \dots + d_n e^{i\gamma_n}| = |d_1 e^{i\gamma_1^0} + \dots + d_n e^{i\gamma_n^0}|. \tag{1.4}$$

The extremal property of the point $(\gamma_1^0, \dots, \gamma_n^0)$ in (1.4) implies that the expression $\Big(\sum\limits_{k=1}^n d_k e^{i\gamma_k^0}\Big) e^{-i\gamma_j^0}$ is real for all $j = 1, \dots, n$. By the same token the problem can be reduced to the considered case $n = 2$ for the sum of two real numbers. Therefore (1.3) is valid for all n.

b) (1.3) $\Longrightarrow$ (1.2). Suppose this is not valid for a certain integer $n \ge 2$. Then $\varepsilon > 0$ and numbers $l \in \{1, \dots, n\}$ and $p \in \{0, \dots, n-2\}$ exist such that

$$d_l + d_{l+1} + \dots + d_{l+p} = \frac{1+\alpha}{2} + \frac{\varepsilon}{2}, \quad d_{l+p+1} + \dots + d_{l+n-1} = \frac{1-\alpha}{2} - \frac{\varepsilon}{2}. \tag{1.5}$$

In condition (1.3) with no loss of generality we can assume that $0 \le \gamma_l \le \gamma_{l+1} \le \dots \le \gamma_{l+n-1} \le 2\pi$, where $\gamma_{l+n} = \gamma_l$. Therefore we can take $\gamma_l = \gamma_{l+1} = \dots = \gamma_{l+p} = 0$, $\gamma_{l+p+1} = \dots = \gamma_{l+n-1} = \pi$. Then from (1.5) we get $\sum\limits_{k=1}^n d_k e^{i\gamma_k} = \alpha + \varepsilon$. This contradicts condition (1.3) and thus proves Lemma 1. □

Remark 1. Condition (1.2) in Lemma 1 is equivalent to the following condition: For all natural $l \in [1, n]$, $p \in [0, n-1]$,

$$|d_l + d_{l+1} + \dots + d_{l+p}| + |d_{l+p+1} + \dots + d_{l+n-1}| \le \alpha \tag{1.6}$$

(for $p = n-1$, inequality (1.6) takes the form $|d_l + d_{l+1} + \dots + d_{l+n-1}| \le \alpha$).

Indeed, it suffices to prove inequality (1.6) for $p \in [0, n-2]$. We denote $d_l + d_{l+1} + \dots + d_{l+p} = x$. Then from condition (1.1) we obtain $d_{l+p+1} + \dots + d_{l+n-1} = 1 - x$. The fulfillment of inequality (1.6) means that $|x| + |1 - x| \le \alpha$; this inequality is equivalent to condition (1.2).

As the limit case of Lemma 1, we obtain

Corollary 1. Let μ be a real function of bounded variation on $[0, 2\pi)$, satisfying the first of conditions (0.3). Let us continue μ to $\mathbb{R}$: $\mu(2\pi + t) = \mu(2\pi - 0) + \mu(t)$. Then the following two conditions are equivalent:

$$\text{if } \; a < b \le 2\pi + a, \quad \text{then } \; \frac{1-\alpha}{2} \le \int_a^b d\mu(t) \le \frac{1+\alpha}{2} \quad \forall\, a, b \in \mathbb{R}; \tag{1.7}$$

$$\left| \int_0^{2\pi} e^{it} d\mu(t) \right| \le \alpha. \tag{1.8}$$

Definition 1. Denote by $\mathcal{I}_\alpha(R)$, $\alpha \geq 1$, a set of all real functions of bounded variation on $[0, 2\pi)$, which satisfy both (0.3) and condition (1.7).

Definition 2. We denote by $\mathcal{U}_\alpha(R)$, $\alpha \geq 1$, the closure in the topology of uniform convergence inside the disk Δ of the set of functions $f(z) = z + \cdots$ of the form (0.2) with $\mu \in \mathcal{I}_\alpha$.

Obviously, $\mathcal{U}_{\alpha_1}(R) \subset \mathcal{U}_{\alpha_2}(R)$ if $\alpha_1 \leq \alpha_2$.

Theorem 1. *For any $\alpha \geq 1$, the family $\mathcal{U}_\alpha(R)$ is an l. i. f. of order α.*

Proof. Let $f \in \mathcal{U}_\alpha(R)$; then from Definition 2 we obtain $f' = \lim\limits_{n\to\infty} f_n'$, where

$$f_n'(z) = \exp\left[-2\int_0^{2\pi} \log(1 - ze^{it})d\mu_n(t)\right], \quad z \in \Delta, \quad \mu_n \in \mathcal{I}_\alpha(R).$$

a) Let us first show that $\mathcal{U}_\alpha(R)$ is an l. i. f. Consider the function $\psi_n = \Lambda_\phi[f_n]$, where

$$\phi(z) = e^{i\theta}\frac{z + a}{1 + \overline{a}z}, \quad a \in \Delta, \quad \theta \in \mathbb{R},$$

is a fixed conformal automorphism of Δ. From (0.1) it follows

$$\psi_n'(z) = \frac{f_n'\left(e^{i\theta}\frac{z+a}{1+\overline{a}z}\right)}{f_n'(e^{i\theta}a)(1 + \overline{a}z)^2} = \exp\left[-2\int_0^{2\pi} \log\left(1 - z\frac{e^{i(\theta+t)} - \overline{a}}{1 - e^{i(\theta+t)}a}\right) d\mu_n(t)\right]. \tag{1.9}$$

Let

$$e^{i\chi(t)} = \frac{e^{i(\theta+t)} - \overline{a}}{1 - e^{i(\theta+t)}a}, \quad t \in [0, 2\pi). \tag{1.10}$$

With the growth of t from 0 to 2π the corresponding values $\chi(t)$ increase from χ_0 to $\chi_0 + 2\pi$. Therefore relation (1.10) defines a real function $t = t(\chi)$ which sends the half-segment $[\chi_0, \chi_0 + 2\pi)$ to $[0, 2\pi)$. Then from (1.9) we obtain

$$\psi_n'(z) = \exp\left[-2\int_{\chi_0}^{\chi_0+2\pi} \log(1 - ze^{i\chi})d\mu_n(t(\chi))\right],$$

and the function $\nu_n(\chi) = \mu_n(t(\chi))$ satisfies condition (0.3)

$$\int_0^{2\pi} d\nu_n(\chi) = \int_{\chi_0}^{\chi_0+2\pi} d\nu_n(\chi) = \int_0^{2\pi} d\mu_n(t) = 1.$$

For $a \leq \chi \leq b < a + 2\pi$, we have

$$\int_a^b d\nu_n(\chi) = \int_{t(a)}^{t(b)} d\mu_n(t), \quad t(a) \leq t(b) < t(a) + 2\pi.$$

Since $\mu_n \in \mathcal{I}_\alpha(R)$, we have $\frac{1-\alpha}{2} \leq \int\limits_{t(a)}^{t(b)} d\mu_n(t) \leq \frac{1+\alpha}{2}$ by condition (1.7). Consequently, ν_n also satisfies condition (1.7), i. e., $\nu_n \in \mathcal{I}_\alpha(R)$. Therefore

$$\psi_n'(z) = \exp\left[-2\int_{\chi_0}^{\chi_0+2\pi} \log(1 - ze^{i\chi})d\nu_n(\chi)\right] = \exp\left[-2\int_0^{2\pi} \log(1 - ze^{i\chi})d\nu_n(\chi)\right],$$

and $\psi_n \in \mathcal{U}_\alpha(R)$ for all natural n.

From the existence of the uniform inside Δ limit $\lim\limits_{n\to\infty} f_n'(z) = f'(z)$ it follows (see (1.9)) that for any fixed automorphism ϕ the uniform inside Δ limit exists $\lim\limits_{n\to\infty} \psi_n(z) = \Lambda_\phi[f](z) = \psi(z)$. The closedness of the family $\mathcal{U}_\alpha(R)$ implies that $\psi = \Lambda_\phi[f] \in \mathcal{U}_\alpha(R)$. Consequently, $\mathcal{U}_\alpha(R)$ is an l. i. f.

b) Now let us show that $\operatorname{ord}\mathcal{U}_\alpha(R)=\alpha$. As is known (see ([1], p. 115), for the l. i. f. $\mathfrak{M}$ we have

$$\operatorname{ord}\mathfrak{M}=\sup_{f\in\mathfrak{M},\ z\in\Delta}\left|\frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)}-\overline{z}\right|.$$

Therefore it suffices to prove the inequality $\left|\frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)}-\overline{z}\right|\le\alpha$ for all $z\in\Delta$ and for any function $f\in\mathcal{U}_\alpha(R)$ possessing representation (0.2). From (0.2) with a fixed $z=re^{i\theta}$ we obtain

$$\left|\frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)}-\overline{z}\right|=\left|\int_0^{2\pi}\frac{(1-r^2)e^{i(t+\theta)}}{1-re^{i(t+\theta)}}d\mu(t)-r\right|=\left|\int_0^{2\pi}e^{i\chi(t)}d\mu(t)\right|, \tag{1.11}$$

where $e^{i\chi(t)}=\frac{e^{i(\theta+t)}-r}{1-re^{i(\theta+t)}}$, $t\in[0,2\pi)$. Arguments similar to those in the first part of the proof and with the same notation show that the last expression in (1.11) can be rewritten as follows

$$\left|\int_{\chi_0}^{\chi_0+2\pi}e^{i\chi}d\mu(t(\chi))\right|=\left|\int_{\chi_0}^{\chi_0+2\pi}e^{i\chi}d\nu(\chi)\right|=\left|\int_0^{2\pi}e^{i\chi}d\nu(\chi)\right|.$$

As above, we can prove that $\nu(\chi)\in\mathcal{I}_\alpha(R)$. Then from (1.8) it follows that expression in (1.11) does not exceed α. Thus, $\operatorname{ord}\mathcal{U}_\alpha(R)\le\alpha$.

The example of the function

$$k_\alpha(z)=\frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^\alpha-1\right]=z+\alpha z^2+\cdots\in\mathcal{U}_\alpha(R)$$

(here $1^\alpha=1$, $\alpha\ge1$) demonstrates that $\operatorname{ord}\mathcal{U}_\alpha(R)\ge\alpha$. Thus, $\operatorname{ord}\mathcal{U}_\alpha(R)=\alpha$. □

2. *Family $V^*_{2\alpha}$, its coincidence with $\mathcal{U}_\alpha(R)$.* In [5]–[7] the l. i. f. $\mathcal{U}^*_\alpha$ of order α were introduced and studied. The function f is in $\mathcal{U}^*_\alpha$ (see [7]) if and only if a function $s\in\mathcal{K}$ and a regular in Δ function ω, $\omega(0)=0$, $|\omega(z)|<1$ for $z\in\Delta$ (we denote by Ω the class of such functions) exist such that

$$f'(z)=s'(z)\exp\left[-2\int_0^{2\pi}\log(1-\omega(z)e^{it})d\mu(t)\right],\quad z\in\Delta,\quad f(0)=0, \tag{2.1}$$

where μ is a complex-valued function of bounded variation, satisfying the condition

$$\int_0^{2\pi}d\mu(t)=0,\quad \int_0^{2\pi}|d\mu(t)|\le\alpha-1, \tag{2.2}$$

(we denote by $\mathcal{I}^*_\alpha$ the class of such functions). As is known, $\mathcal{U}^*_1=\mathcal{K}$ and for $\alpha>1$ none of the classes $\mathcal{U}'_\alpha$ and $\mathcal{U}^*_\alpha$ contains another one.

As it was said at the beginning of this article, to the function $f\in V_{2\alpha}$ in its integral representation (0.2) a real function of bounded variation μ corresponds satisfying conditions (0.3). Such a function μ can be written with the help of nondecreasing on $[0,2\pi]$ functions of bounded variation $\mu^{(1)}$, $\mu^{(2)}$: $\mu=\mu^{(1)}-\mu^{(2)}$, $\int\limits_0^{2\pi}d\mu^{(1)}(t)\le\frac{\alpha+1}{2}$, $\int\limits_0^{2\pi}d\mu^{(2)}(t)\le\frac{\alpha-1}{2}$. In its turn, we always can write $\mu^{(1)}=\mu^{(1)}_0+\mu^{(1)}_*$, where $\mu^{(1)}_0$, $\mu^{(1)}_*$ also do not decrease and possess complete variations on $[0,2\pi]$, which are equal to 1 and $(\alpha-1)/2$, respectively. Therefore, for every function $f\in V_{2\alpha}$, a function $s\in\mathcal{K}$ and a real function of bounded variation $\mu_*=\mu^{(1)}_*-\mu^{(2)}$, which satisfies conditions (2.2), exist such that f' is defined by formula (2.1), where $\omega(z)\equiv z$, $\mu=\mu_*$. A set of all functions of that sort μ_* is denoted by $\mathcal{I}^*_\alpha(R)$. Obviously, $\mathcal{I}^*_\alpha(R)$ consists of real functions μ_* from $\mathcal{I}^*_\alpha$.

Remark 2. Starting with a given function $f\in V_{2\alpha}$ in expression (2.1) ($\omega(z)\equiv z$), the above functions $\mu=\mu_*$ and s are determined, in the general case, not uniquely.

Definition 3. Let $\alpha \ge 1$. We denote

$$V_{2\alpha}^* = \{f : f(0) = 0,\ f' \text{ has the form (2.1)},\ s \in \mathcal{K},\ \omega \in \Omega,\ \mu \in \mathcal{I}_\alpha^*(R)\}.$$

Obviously, $V_{2\alpha} \subset V_{2\alpha}^* \subset \mathcal{U}_\alpha^*$. In what follows we will prove that $V_{2\alpha}^* = \mathcal{U}_\alpha(R)$.

For $\gamma \in (-\pi/2, \pi/2)$, we denote by P_γ a set of all analytic in Δ functions $p(z) = 1 + \cdots$ such that $\operatorname{Re}\{e^{i\gamma}p(z)\} > 0$ in Δ. Let $P_{(\pi)} = \bigcup\limits_{\gamma \in (-\pi/2,\pi/2)} P_\gamma$.

Lemma 2 (see [8]). *If $\gamma \in (-\pi/2, \pi/2)$ and the function p has the form*

$$p(z) = 1 + \cos\gamma\Big(\sum_{k=1}^n \lambda_k \frac{1+\sigma_k z}{1-\sigma_k z} - 1\Big), \quad \sum_{k=1}^n \lambda_k = 1, \quad \lambda_k \ge 0, \quad \sigma_k \in \partial\Delta, \tag{2.3}$$

$k = 1, \dots, n$, *then $p \in P_{(\pi)}$ and it can be written as follows*

$$p(z) = \frac{(1-ze^{it_1})(1-ze^{it_3})\dots(1-ze^{it_{2m-1}})}{(1-ze^{it_2})(1-ze^{it_4})\dots(1-ze^{it_{2m}})}, \quad m \le n, \tag{2.4}$$

$t_1 < t_2 < t_3 < \cdots < t_{2m} < t_1 + 2\pi$ *(or $t_2 < t_1 < t_4 < t_3 < \cdots < t_{2m} < t_{2m-1} < t_2 + 2\pi$). Vice versa, if p has the form (2.4), then $p \in P_{(\pi)}$ and it can be written in the form (2.3).*

Lemma 3. *If $f \in \mathcal{U}_\alpha(R)$, $\alpha > 1$, and in the integral representation* (0.2) *to this function the step-function μ corresponds with discontinuities at $n \ge 2$ points $t_1, \dots, t_n$ and the respective gaps $d_1, \dots, d_n$, not all of them being positive, then we can write f' as follows*

$$f'(z) = (p(z))^{2\delta}(f^*)'(z),$$

where $\delta = |d_{k_0}| = \min\limits_{k \in [1,n]} |d_k|$, $p \in P_{(\pi)}$, $f^* \in \mathcal{U}_{\alpha-2\delta}(R)$ and to this function f^* in its integral representation (0.2) *the step-function μ^* corresponds possessing at most $(n-1)$ discontinuities.*

Note that by inequality (1.2) we have $\delta \le (\alpha-1)/2$; therefore $\alpha - 2\delta \ge 1$.

Proof. The numbers $d_1, \dots, d_n$ satisfy the condition $\frac{1-\alpha}{2} \le d_k \le \frac{1+\alpha}{2}$, which follows from (1.2), since $\mu \in \mathcal{I}_\alpha(R)$. Let us select among the numbers $d_1, \dots, d_n$ $2N$ consecutive groups of neighboring numbers of the same sign

$$\underbrace{d_1, \dots, d_{k_1}}_{\mathcal{D}_1},\quad \underbrace{d_{k_1+1}, \dots, d_{k_2}}_{\mathcal{D}_2}, \dots, \underbrace{d_{k_{2N-1}+1}, \dots, d_{k_{2N}}}_{\mathcal{D}_{2N}}$$

(their number is always even). Denote $\delta = |d_{k_0}| = \min\limits_{k \in [1,n]} |d_k|$. In every of the groups $\mathcal{D}_1, \dots, \mathcal{D}_{2N}$ we choose a number (in the group containing d_{k_0} we choose namely d_{k_0}) and reduce the module of the chosen number by δ. This operation can be called a reduction on δ. In that newly obtained set of numbers, at most $(n-1)$ differ from zero. Keeping the same order of succession, we denote by $d_1^*, \dots, d_n^*$ the numbers of the new set. Obviously, $\sum\limits_{l=1}^n d_l^* = 1$ (the number of groups $\mathcal{D}_k$ is even). Let us show that, for any integer $(p+1)$, $l \in [1, n]$, we have

$$\frac{1-\alpha}{2} + \delta \le d_l^* + \cdots + d_{l+p}^* \le \frac{1+\alpha}{2} - \delta; \tag{2.5}$$

here we use the notation $d_{n+j}^* = d_j^*$ for all $j = 1, \dots, n$. Let us first prove the left side of inequality (2.5).

Consider the set $d_l^*, \dots, d_{l+p}^*$. If this will be necessary we can delete from this set a part of positive numbers from one edge of the set (for example, from the left edge) and extend this set from other edge (the right edge) by addition of negative numbers so that the resulting set be a consequence of the reduction on δ of a certain consecutive set of numbers $\{d_k\}$, which is a consecutive union of L

successive groups $\{\mathcal{D}_\nu\}_{\nu=\nu_0}^{\nu_0+L}$. As a result, we obtain the new set of numbers $\{d^*_\rho\}_{\rho=\lambda}^{\lambda+\beta}$, whose sum is $\sum\limits_{\rho=\lambda}^{\lambda+\beta} d^*_\rho \le d^*_l + \dots + d^*_{l+p}$. The three cases are possible.

1) L is even. Then $\sum\limits_{\rho=\lambda}^{\lambda+\beta} d^*_\rho = \sum\limits_{\rho=\lambda}^{\lambda+\beta} d_\rho$, since the last sum is equal to the sum of all numbers in all groups of numbers $\mathcal{D}_{\nu_0}, \dots, \mathcal{D}_{\nu_0+L}$. In this set $\{\mathcal{D}_\nu\}_{\nu=\nu_0}^{\nu_0+L}$ one of the edge groups (we can assume that this group is $\mathcal{D}_{\nu_0}$) consists of only negative numbers. Therefore $\mathcal{D}_{\nu_0+L+1}$ also consists only of negative numbers whose sum is not lesser than $-\delta$. The sum of all numbers in the groups $\mathcal{D}_{\nu_0}, \dots, \mathcal{D}_{\nu_0+L+1}$ is not lesser than $\frac{1-\alpha}{2}$ by inequality (1.2). Consequently,

$$\sum_{\rho=\lambda}^{\lambda+\beta} d_\rho - \delta \ge \frac{1-\alpha}{2},$$

and we have proved the left side of inequality (2.5) for this case.

2) If in $\{\mathcal{D}_\nu\}_{\nu=\nu_0}^{\nu_0+L}$ the first and the last groups consist of only negative numbers, then $\sum\limits_{\rho=\lambda}^{\lambda+\beta} d^*_\rho = \sum\limits_{\rho=\lambda}^{\lambda+\beta} d_\rho + \delta$; by inequality (1.2) we have $\sum\limits_{\rho=\lambda}^{\lambda+\beta} d_\rho \ge \frac{1-\alpha}{2}$. Hence we obtain the left-hand side of inequality (2.5).

3) If in $\{\mathcal{D}_\nu\}_{\nu=\nu_0}^{\nu_0+L}$ the first and last groups consist of only positive numbers, then $\sum\limits_{\rho=\lambda}^{\lambda+\beta} d^*_\rho = \sum\limits_{\rho=\lambda}^{\lambda+\beta} d_\rho - \delta$. The sum of numbers in the group $\mathcal{D}_{\nu_0}$ is not less than δ, the same can be said about the group $\mathcal{D}_{\nu_0+L}$. By inequality (1.2) the sum of all numbers in the groups $\{\mathcal{D}_\nu\}_{\nu_0+1}^{\nu_0+L-1}$ is not less than $\frac{1-\alpha}{2}$. Consequently,

$$\sum_{\rho=\lambda}^{\lambda+\beta} d_\rho \ge 2\delta + \frac{1-\alpha}{2} \Longrightarrow \sum_{\rho=\lambda}^{\lambda+\beta} d^*_\rho \ge \delta + \frac{1-\alpha}{2}.$$

By the same token we have proved the left-hand side of inequality (2.5). In a similar way one can prove the right-hand side of this inequality.

The facts proved above imply that f' can be written in the form $f'(z) = (p_N(z))^{2\delta}(f^*)'(z)$; here $p_N(z)$ is from (2.4) for $m = N$, τ_ν is one of the discontinuity points of the function μ, to which corresponds the jump d_ν from the group $\mathcal{D}_\nu$, $f^* \in \mathcal{U}_{\alpha-2\delta}(R)$, and to this function f^* in its integral representation (0.2) the step-function μ^* corresponds possessing at most $(n-1)$ discontinuities. By Lemma 2 we have $p_N(z) \in P_{(\pi)}$. This proves Lemma 3. □

Theorem 2. *$\mathcal{U}_\alpha(R) = V^*_{2\alpha}$ for all $\alpha \ge 1$.*

Proof. a) Let $f \in \mathcal{U}_\alpha(R)$. Then by Definition 2 we have $f = \lim\limits_{m\to\infty} f_m$ (the limit is uniform inside Δ) and

$$f'_m(z) = \exp\Big[-2\int_0^{2\pi} \log(1 - ze^{it})\, d\mu_m(t)\Big], \quad \mu_m \in \mathcal{I}_\alpha(R). \tag{2.6}$$

Every function μ_m can be approximated uniformly inside $[0, 2\pi]$ by the step-functions $\{\mu^k_m\}_{k=1}^\infty$; in doing so, we can assume that $\mu^k_m(0) = \mu_m(0)$, $\mu^k_m(2\pi) = \mu_m(2\pi)$ (consequently, $\int\limits_0^{2\pi} d\mu^k_m(t) = 1$ for all k and m). Therefore, for all $k \in \mathbb{N}$, a step-function $\nu^k_m(t)$, $t \in [0, 2\pi]$, exists such that its discontinuity points coincide with the discontinuity points of μ^k_m, the values at the discontinuity points coincide with the values at these points of the function μ_m, $\int\limits_0^{2\pi} d\nu^k_m(t) = 1$, $(\mu^k_m(t) - \nu^k_m(t)) \xrightarrow[k\to\infty]{} 0$

uniformly on $[0, 2\pi]$. Hence and from the fact that $\mu_m \in \mathcal{I}_\alpha(R)$ it follows that $\nu_m^k \in \mathcal{I}_\alpha(R)$, because ν_m^k satisfy condition (1.7). The uniform convergence $\nu_m^k \xrightarrow[k\to\infty]{} \mu_m$ implies that in (2.6) the functions $\mu_m \in \mathcal{I}_\alpha(R)$ can be treated as the step-functions.

Let us fix m. By Lemma 3 the derivative of the function f_m can be written in the form $f_m'(z) = (p_1(z))^{2\delta_1}(f^*)'(z)$, where $p_1 \in P_{(\pi)}$, $\delta_1 \in [0, (\alpha-1)/2]$, $f^* \in \mathcal{U}_{\alpha-2\delta_1}(R)$. In its turn, by Lemma 3, we have $(f^*)'(z) = (p_2(z))^{2\delta_2}(h^*)'(z)$, where $p_2 \in P_{(\pi)}$, $\delta_2 \in [0, (\alpha-1-2\delta_1)/2]$, $h^* \in \mathcal{U}_{\alpha-2\delta_1-2\delta_2}(R)$. Within a finite number of steps we arrive at the conclusion

$$f_m'(z) = (p_1(z))^{2\delta_1}\dots(p_k(z))^{2\delta_k}s_m'(z), \quad \delta(m) = \sum_{j=1}^k \delta_j \in [0, (\alpha-1)/2],$$

where $s_m \in \mathcal{U}_{\alpha-\delta(m)}(R)$, $p_j \in P_{(\pi)}$, $\delta_j \ge 0$, $j = 1, \dots, k$, and in integral representation (0.2) to function s_m' the step-function $\mu^*(t)$ corresponds not possessing negative jumps. Therefore we have $s_m(z) \in \mathcal{K}$. From the definition of the class $P_{(\pi)}$ it follows that $\prod_{j=1}^k (p_j(z))^{\beta_j} \in P_{(\pi)}$ if $\sum_{j=1}^k \beta_j = 1$, $\beta_j \ge 0$. Therefore $f_m' = (q_m(z))^{2\delta(m)}s_m'(z)$, where $q_m \in P_{(\pi)}$. The closedness of the classes $P_{(\pi)}$ and $\mathcal{K}$ in the topology of uniform convergence inside Δ implies

$$f'(z) = \lim_{m\to\infty} f_m'(z) = p^{2\delta}(z)s'(z), \quad p \in P_{(\pi)}, \quad s \in \mathcal{K}, \quad \delta \in \left[0, \frac{\alpha-1}{2}\right].$$

Since $p \in P_{(\pi)}$, $\gamma \in (-\pi/2, \pi/2)$ exists such that $\operatorname{Re}\{e^{i\gamma}p(z)\} > 0$, $z \in \Delta$. By choosing $\sigma, \eta \in \partial\Delta$ so that $\operatorname{Re}\{e^{i\gamma}\frac{1-z\sigma}{1-z\eta}\} > 0$, we write $p(z) = \frac{1-\omega(z)\sigma}{1-\omega(z)\eta}$, $\omega \in \Omega$. Consequently, we can write

$$f'(z) = s'(z)\exp\left[-2\int_0^{2\pi} \log(1-\omega(z)e^{it})d\mu_*(t)\right], \quad \int_0^{2\pi} d\mu_*(t) = 0;$$

here μ_* is a step-function with two discontinuities $(\pm\delta)$, i. e., $\mu_* \in \mathcal{I}_\alpha^*(R)$ and $f \in V_{2\alpha}^*$.

b) Vice versa, let $f \in V_{2\alpha}^*$. Then (see Definition 3) $s \in \mathcal{K}$, $\mu \in \mathcal{I}_\alpha^*(R)$, $\omega \in \Omega$ exist such that

$$\psi^{\alpha-1}(z) = \exp\left[-2\int_0^{2\pi} \log(1-ze^{it})d\mu(t)\right] \text{ and } f'(z) = s'(z)\psi^{\alpha-1}(\omega(z)), \quad z \in \Delta.$$

Here the function μ can be written as follows $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are increasing functions on $[0, 2\pi]$, whose complete variation does not exceed $(\alpha-1)/2$ (see (2.2)). Therefore convex functions $s_1, s_2 \in \mathcal{K}$ exist such that $\psi^{\alpha-1}(z) = (s_1'(z)/s_2'(z))^{(\alpha-1)/2}$. Since s_1', s_2' are derivatives of convex functions, uniformly inside Δ they can be approximated by the functions (see, e. g., [9], corollary 1.1):

$$s_{1,n}'(z) = \prod_{k=1}^n (1-z\sigma_k)^{-2\delta_k} \quad \text{and} \quad s_{2,n}'(z) = \prod_{k=1}^n (1-z\eta_k)^{-2\lambda_k},$$

respectively. Here $\sigma_k, \eta_k \in \partial\Delta$, $n \in \mathbb{N}$, $\sum_{k=1}^n \delta_k = \sum_{k=1}^n \lambda_k = 1$. Consequently, uniformly inside Δ we have

$$\psi_n(z) = \prod_{k=1}^n \frac{(1-z\eta_k)^{\lambda_k}}{(1-z\sigma_k)^{\delta_k}} \xrightarrow[n\to\infty]{} \psi(z).$$

Since the last product can be written as follows

$$\prod_{j=1}^N \left(\frac{1-z\eta_j'}{1-z\sigma_j'}\right)^{\xi_j}, \quad N \in \mathbb{N}, \quad \eta_j', \sigma_j' \in \partial\Delta, \quad \sum_{j=1}^N \xi_j = 1, \quad \xi_j \ge 0 \text{ for all } j,$$

and for any j the function $\frac{1-z\eta_j'}{1-z\sigma_j'}$ is in $P_{(\pi)}$, we also have $\frac{1-\omega(z)\eta_j'}{1-\omega(z)\sigma_j'} \in P_{(\pi)}$. However, in that case, $\psi_n(\omega(z)) \in P_{(\pi)}$, and therefore we have as well $\psi(\omega(z)) \in P_{(\pi)}$. Consequently, the function

$\psi(\omega(z))$ can be written as the uniform inside Δ limit $\lim_{n\to\infty} p_n(z)$, where $p_n \in P_\theta$ with a certain $\theta \in (-\pi/2, \pi/2)$ and

$$p_n(z) = 1 + \cos\theta\Big(\sum_{k=1}^{n} \lambda_k \frac{1+ze^{i\gamma_k}}{1-ze^{i\gamma_k}} - 1\Big), \quad \sum_{k=1}^{n}\lambda_k = 1, \quad \lambda_k \ge 0, \quad \gamma_k \in \mathbb{R},$$

$k = 1, \dots, n$. Using Lemma 2, we rewrite p_n in the form (2.4) with $m = N \le n$. Since the derivative of the convex function s can be approximated uniformly inside Δ by the functions

$$r_n'(z) = \prod_{k=1}^{n}(1-ze^{i\tau_k})^{-2\varepsilon_k}, \quad \sum_{k=1}^{n}\varepsilon_k = 1, \quad \varepsilon_k \ge 0, \quad \tau_k \in \mathbb{R}, \quad k = 1, \dots, n,$$

the uniform inside Δ limit exists: $\lim_{n\to\infty}[r_n'(z)p_n^{\alpha-1}(z)] = f'(z)$. Here the expression after the symbol of limit can be written with the use of the Stieltjes integral

$$r_n'(z)p_n^{\alpha-1}(z) = \exp\Big[-2\int_0^{2\pi}\log(1-ze^{it})\,d\mu_n(t)\Big],$$

where μ_n is the step-function satisfying conditions (1.7) and the first of conditions (0.3), i.e., $\mu_n \in \mathcal{I}_\alpha(R)$. Consequently, $f \in \mathcal{U}_\alpha(R)$. □

From Theorems 1 and 2 we obtain

Corollary 2. $V_{2\alpha}^*$ is an l.i.f. of order α.

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Petrozavodsk State University
Lodz State University