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ON THE LOCUS OF p-CHARACTERS DEFINING SIMPLE REDUCED ENVELOPING ALGEBRAS

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Abstract

We confirm in two cases the conjecture stating that the reduced enveloping algebra $U_\xi(\mathfrak{g})$ of a restricted Lie algebra $\mathfrak{g}$ is simple if and only if the alternating bilinear form associated with the given p-character $\xi \in \mathfrak{g}^*$ is nondegenerate.

Key words: restricted Lie algebras, solvable Lie algebras, Frobenius Lie algebras, reduced enveloping algebras.

In the representation theory of a finite dimensional p-Lie algebra $\mathfrak{g}$ over an algebraically closed field k of characteristic $p > 0$ one is naturally led to consider the family of reduced enveloping algebras $U_\xi(\mathfrak{g})$ associated with linear functions $\xi \in \mathfrak{g}^*$ (see [1]). The algebra $U_\xi(\mathfrak{g})$ is defined as the factor algebra of the universal enveloping algebra $U(\mathfrak{g})$ by its ideal generated by central elements $x^p - x^{[p]} - \xi(x)^p \cdot 1$ with $x \in \mathfrak{g}$, and ξ is called the p-character of any $\mathfrak{g}$-module which can be realized as a module over $U_\xi(\mathfrak{g})$. There is a certain, still far from fully understood, relation between generic properties of the family of reduced enveloping algebras and generic properties of the family of stabilizers of linear functions. The stabilizer $\mathfrak{z}(\xi)$ of $\xi \in \mathfrak{g}^*$ coincides with the radical of the alternating bilinear form $\beta_\xi : \mathfrak{g} \times \mathfrak{g} \to k$ defined by the rule

$$\beta_\xi(x, y) = \xi\big([x, y]\big) \quad \text{for } x, y \in \mathfrak{g}.$$

The Lie algebra $\mathfrak{g}$ is called Frobenius if β_ξ is nondegenerate for at least one ξ.

In general one cannot determine the type of one particular algebra $U_\xi(\mathfrak{g})$ just knowing $\mathfrak{z}(\xi)$. It is quite interesting and surprising that sometimes this can be done. In [2] it was conjectured that $U_\xi(\mathfrak{g})$ is simple if and only if $\mathfrak{z}(\xi) = 0$, that is, if and only if β_ξ is nondegenerate. The purpose of the present article is to verify this conjecture in two cases. When $\mathfrak{g}$ is solvable and $p > 2$ we do this using the description of irreducible $\mathfrak{g}$-modules due to Strade [3]. We have to make more careful selections of subalgebras from which irreducible $\mathfrak{g}$-modules are obtained by induction. The second case occurs when $\mathfrak{g}$ is Frobenius and all adjoint derivations of $\mathfrak{g}$ lie in the Lie algebra of the automorphism group. Here we apply geometric arguments to the extension of the family of reduced enveloping algebras constructed in [4].

An example at the end of the paper shows that semisimplicity of the algebra $U_\xi(\mathfrak{g})$ cannot be recognized in terms of $\mathfrak{z}(\xi)$ by means of a possible generalization of the above conjecture.

1. Solvable Lie algebras

It is assumed in this section that $\mathfrak{g}$ is solvable and $p > 2$. Recall that a polarization of $\mathfrak{g}$ at $\xi \in \mathfrak{g}^*$ is a Lie subalgebra which is simultaneously a maximal totally isotropic subspace with respect to the alternating bilinear form β_ξ [5].

Denote by $\mathcal{P}$ the set of all triples $(\mathfrak{p}, \mathfrak{a}, \lambda)$ such that $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{g}$ are vector subspaces, $\lambda \in \mathfrak{a}^*$ is a linear function and there exists a chain of subspaces

$$0 = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \ldots \subset \mathfrak{a}_n = \mathfrak{a} \subset \mathfrak{p} = \mathfrak{p}_n \subset \ldots \subset \mathfrak{p}_1 \subset \mathfrak{p}_0 = \mathfrak{g} \tag{1}$$

with the property that

$$[\mathfrak{p}_{i-1}, \mathfrak{a}_i] \subset \mathfrak{a}_i \quad \text{and} \quad \mathfrak{p}_i = \{x \in \mathfrak{p}_{i-1} \mid \lambda([x, \mathfrak{a}_i]) = 0\} \tag{2}$$

for all $i = 1, \ldots, n$. As one checks by induction on i, each $\mathfrak{p}_i$ is a p-subalgebra of $\mathfrak{g}$, and $\mathfrak{a}_i$ is an ideal of $\mathfrak{p}_{i-1}$. In particular, $\mathfrak{p}$ is a p-subalgebra of $\mathfrak{g}$, and $\mathfrak{a}$ is an ideal of $\mathfrak{p}$. Furthermore, λ vanishes on $[\mathfrak{p}, \mathfrak{a}]$ and, therefore, also on $[\mathfrak{a}, \mathfrak{a}]$.

Lemma 1. *Suppose that* $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}$. *If* $\xi \in \mathfrak{g}^*$ *is a linear function such that* $\lambda(x)^p - \lambda(x^{[p]}) = \xi(x)^p$ *for all* $x \in \mathfrak{a}$ *and* W *is an irreducible* $U_\xi(\mathfrak{p})$*-module such that* $xw = \lambda(x)w$ *for all* $x \in \mathfrak{a}$ *and* $w \in W$, *then the induced* $\mathfrak{g}$*-module* $U_\xi(\mathfrak{g}) \otimes_{U_\xi(\mathfrak{p})} W$ *is irreducible.*

Here $U_\xi(\mathfrak{p})$ stands for the reduced enveloping algebra of $\mathfrak{p}$ corresponding to the restriction of ξ to $\mathfrak{p}$. The proof is obtained by a repeated application of the characteristic p analog of Blattner's irreducibility criterion [6, Theorem 3].

We will need additional conditions on triples. Denote by $\mathcal{P}'$ the set of all triples $(\mathfrak{p}, \mathfrak{a}, \lambda)$ such that $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{g}$ are vector subspaces, $\lambda \in \mathfrak{a}^*$ is a linear function, and there exists a chain of subspaces

$$0 = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \ldots \subset \mathfrak{a}_n = \mathfrak{a} \subset \mathfrak{p} \subset \widetilde{\mathfrak{p}}_n \subset \ldots \subset \widetilde{\mathfrak{p}}_1 \subset \widetilde{\mathfrak{p}}_0 = \mathfrak{g} \tag{3}$$

with the property that

$$[\widetilde{\mathfrak{p}}_{i-1}, \mathfrak{a}_i] \subset \mathfrak{a}_i, \tag{4}$$

$$\widetilde{\mathfrak{p}}_i = \{x \in \widetilde{\mathfrak{p}}_{i-1} \mid \lambda([x, \mathfrak{a}_i']) = 0\}, \quad \text{where} \quad \mathfrak{a}_i' = \{y \in \mathfrak{a}_i \mid \lambda(y) = 0\}, \tag{5}$$

$$\mathfrak{p} = \{x \in \widetilde{\mathfrak{p}}_n \mid \lambda([x, \mathfrak{a}]) = 0\} \tag{6}$$

for all $i = 1, \ldots, n$. We will say that chain (3) is $(\mathfrak{p}, \mathfrak{a}, \lambda)$-admissible in this case.

Lemma 2. *In a* $(\mathfrak{p}, \mathfrak{a}, \lambda)$*-admissible chain each* $\widetilde{\mathfrak{p}}_i$ *is a* p*-subalgebra,* $\mathfrak{a}_i$ *is an ideal of* $\widetilde{\mathfrak{p}}_{i-1}$, *and* $\mathfrak{a}_i'$ *is an ideal of* $\widetilde{\mathfrak{p}}_i$. *Furthermore,* $\mathfrak{p}$ *is an ideal of* $\widetilde{\mathfrak{p}}_n$.

Proof. Since $[\widetilde{\mathfrak{p}}_i, \mathfrak{a}_i] \subset \mathfrak{a}_i$ by (4) and λ vanishes on $[\widetilde{\mathfrak{p}}_i, \mathfrak{a}_i']$ by (5), we deduce that $[\widetilde{\mathfrak{p}}_i, \mathfrak{a}_i'] \subset \mathfrak{a}_i'$. Since the normalizer of $\mathfrak{a}_i'$ in $\mathfrak{g}$ is a p-subalgebra, an induction on i shows that so too is $\widetilde{\mathfrak{p}}_i$. Now $[\mathfrak{p}, \mathfrak{a}] \subset \mathfrak{a}$ and λ vanishes on $[\mathfrak{p}, \mathfrak{a}]$ by (4) and (6), whence $[\mathfrak{p}, \mathfrak{a}] \subset \mathfrak{a}_n'$. It follows $[[\widetilde{\mathfrak{p}}_n, \mathfrak{p}], \mathfrak{a}] \subset [\widetilde{\mathfrak{p}}_n, \mathfrak{a}_n'] + [\mathfrak{p}, \mathfrak{a}] \subset \mathfrak{a}_n'$, and so $[\widetilde{\mathfrak{p}}_n, \mathfrak{p}] \subset \mathfrak{p}$. □

Lemma 3. *It holds* $\mathcal{P}' \subset \mathcal{P}$.

Proof. Let $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}'$. Consider a $(\mathfrak{p}, \mathfrak{a}, \lambda)$-admissible chain (3) and for each i define $\mathfrak{p}_i = \{x \in \widetilde{\mathfrak{p}}_i \mid \lambda([x, \mathfrak{a}_i]) = 0\}$. We obtain then a chain (1) with $\mathfrak{p}_i \subset \widetilde{\mathfrak{p}}_i$, and it is checked straightforwardly that (2) is fulfilled. Thus $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}$. □

Lemma 4. *Suppose that* $\mathfrak{a}$ *is a one-dimensional ideal of a solvable Lie algebra* $\mathfrak{h}$, *and* $\mathfrak{b}$ *is an ideal of* $\mathfrak{h}$, *minimal with respect to the property that* $\mathfrak{a} \subset \mathfrak{b}$, $\mathfrak{a} \neq \mathfrak{b}$ *and* $[\mathfrak{a}, \mathfrak{b}] = 0$. *Then* $\mathfrak{b}$ *is abelian.*

Proof. Put $\mathfrak{c} = \{x \in \mathfrak{b} \mid [x, \mathfrak{b}] = 0\}$. Then $\mathfrak{c}$ is an ideal of $\mathfrak{h}$ and $\mathfrak{a} \subset \mathfrak{c} \subset \mathfrak{b}$. By the minimality of $\mathfrak{b}$ we have either $\mathfrak{c} = \mathfrak{b}$ or $\mathfrak{c} = \mathfrak{a}$. In the first case $[\mathfrak{b}, \mathfrak{b}] = 0$, and we are done. Suppose that $\mathfrak{c} = \mathfrak{a}$. Then the multiplication in $\mathfrak{b}$ induces a nondegenerate alternating bilinear form $\mathfrak{b}/\mathfrak{a} \times \mathfrak{b}/\mathfrak{a} \to \mathfrak{a}$. In particular, $\mathfrak{b}/\mathfrak{a}$ has even dimension. On the other hand, $\mathfrak{b}/\mathfrak{a}$ is an irreducible $\mathfrak{h}$-module by the minimality of $\mathfrak{b}$, and therefore $\dim \mathfrak{b}/\mathfrak{a}$ is a power of p, hence odd, by [3, Satz 3]. We arrive at a contradiction. □

Lemma 5. *Suppose that* $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}'$. *If* $\mathfrak{a}\ne\mathfrak{p}$, *then there exists a vector subspace* $\mathfrak{b}\subset\mathfrak{p}$ *such that* $\mathfrak{a}$ *is contained in* $\mathfrak{b}$ *properly,* $[\mathfrak{b},\mathfrak{b}]\subset\mathfrak{a}'=\ker\lambda$, *and for every linear function* $\mu\in\mathfrak{b}^*$ *extending* λ *there exists* $\mathfrak{q}$ *satisfying* $(\mathfrak{q},\mathfrak{b},\mu)\in\mathcal{P}'$.

Proof. Consider a $(\mathfrak{p},\mathfrak{a},\lambda)$-admissible chain (3). By Lemma 2 $\mathfrak{a}$ and $\mathfrak{p}$ are ideals of $\widetilde{\mathfrak{p}}_n$. Let us choose an ideal $\mathfrak{b}$ of $\widetilde{\mathfrak{p}}_n$ such that $\mathfrak{a}\subset\mathfrak{b}\subset\mathfrak{p}$, $\mathfrak{a}\ne\mathfrak{b}$, and $\mathfrak{b}$ is minimal with respect to these properties. Then $[\mathfrak{b},\mathfrak{b}]\subset\mathfrak{a}$ since $\widetilde{\mathfrak{p}}_n$ is solvable and $[\mathfrak{b},\mathfrak{a}]\subset\mathfrak{a}'$ by (6). If $\mathfrak{a}\ne\mathfrak{a}'$, then $\dim\mathfrak{a}/\mathfrak{a}'=1$. Lemma 4 applied to the Lie algebra $\widetilde{\mathfrak{p}}_n/\mathfrak{a}'$ and its one-dimensional ideal $\mathfrak{a}/\mathfrak{a}'$ shows that $\mathfrak{b}/\mathfrak{a}'$ is abelian in this case. Thus we have $[\mathfrak{b},\mathfrak{b}]\subset\mathfrak{a}'$ in any case. If $\mu\in\mathfrak{b}^*$ extends λ, then put

$$\widetilde{\mathfrak{p}}_{n+1}=\{x\in\widetilde{\mathfrak{p}}_n\mid\mu([x,\mathfrak{b}'])=0\}\quad\text{and}\quad\mathfrak{q}=\{x\in\widetilde{\mathfrak{p}}_{n+1}\mid\mu([x,\mathfrak{b}])=0\},$$

where $\mathfrak{b}'=\{y\in\mathfrak{b}\mid\mu(y)=0\}$. Note that $\mathfrak{b}\subset\mathfrak{q}$ since μ is zero on $[\mathfrak{b},\mathfrak{b}]\subset\mathfrak{a}'$. Obviously $\mathfrak{q}\subset\widetilde{\mathfrak{p}}_{n+1}\subset\widetilde{\mathfrak{p}}_n$. Setting $\mathfrak{a}_{n+1}=\mathfrak{b}$, we obtain an extension of (3) to a $(\mathfrak{q},\mathfrak{b},\mu)$-admissible chain. Thus $(\mathfrak{q},\mathfrak{b},\mu)\in\mathcal{P}'$. □

We say that $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}$ is maximal if $\mathfrak{p}=\mathfrak{a}$. Denote by $\mathcal{P}_{\max}\subset\mathcal{P}$ the subset of all maximal triples and put $\mathcal{P}'_{\max}=\mathcal{P}_{\max}\cap\mathcal{P}'$. All conclusions of the next proposition with $\mathcal{P}$ in place of $\mathcal{P}'$ were obtained by Strade [3] in a somewhat different language.

Proposition 1. (i) *Given* $\xi\in\mathfrak{g}^*$, *there exists* $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}'_{\max}$ *such that* $\lambda=\xi|_{\mathfrak{p}}$. *In this case* $\mathfrak{p}$ *is a polarization of* $\mathfrak{g}$ *at* ξ.

(ii) *Given an irreducible* $\mathfrak{g}$*-module* V, *there exists* $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}'_{\max}$ *such that the subspace* $V_\lambda=\{v\in V\mid xv=\lambda(x)v$ *for all* $x\in\mathfrak{a}\}$ *is nonzero.*

Proof. Denote by $\mathcal{P}'_\xi\subset\mathcal{P}'$ the subset of those triples $(\mathfrak{p},\mathfrak{a},\lambda)$ for which $\lambda=\xi|_{\mathfrak{p}}$. This subset is nonempty as we may take $\mathfrak{a}=0$, $\mathfrak{p}=\mathfrak{g}$. Suppose that $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}'_\xi$ and $\mathfrak{p}\ne\mathfrak{a}$. Find $\mathfrak{b}$ as in Lemma 5 and set $\mu=\xi|_{\mathfrak{b}}$. There exists $(\mathfrak{q},\mathfrak{b},\mu)\in\mathcal{P}'$ which belongs to $\mathcal{P}'_\xi$ by the choice of μ. We have here $\dim\mathfrak{b}>\dim\mathfrak{a}$. This argument shows that $\mathcal{P}'_\xi\cap\mathcal{P}'_{\max}$ is nonvoid. Indeed, it suffices to pick out $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}'_\xi$ for which $\dim\mathfrak{a}$ is maximal possible. By Lemma 3 $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}$. There exists then a chain (1) satisfying (2). It follows by induction on i that $\mathfrak{p}_i=\{x\in\mathfrak{g}\mid\xi([x,\mathfrak{a}_i])=0\}$. Hence $\mathfrak{p}=\mathfrak{a}$ is a maximal totally isotropic subspace of $\mathfrak{g}$ with respect to β_ξ.

Denote by $\mathcal{P}'_V\subset\mathcal{P}'$ the subset of those triples $(\mathfrak{p},\mathfrak{a},\lambda)$ for which $V_\lambda\ne0$. The triple $(\mathfrak{g},0,0)$ is again in $\mathcal{P}'_V$. Suppose that $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}'_V$ and $\mathfrak{p}\ne\mathfrak{a}$. Let $\mathfrak{b}$ be as in Lemma 5. Since $[\mathfrak{b},\mathfrak{a}]\subset\mathfrak{a}'$, the subspace V_λ is stable under $\mathfrak{b}$. Hence the abelian Lie algebra $\mathfrak{b}/\mathfrak{a}'$ operates in V_λ. It follows that V_λ contains a one-dimensional $\mathfrak{b}$-submodule, say kv. The equality $xv=\mu(x)v$ defines a linear function $\mu\in\mathfrak{b}^*$ which extends λ. We have $v\in V_\mu$ by the construction. Lemma 5 provides a triple $(\mathfrak{q},\mathfrak{b},\mu)\in\mathcal{P}'$ which belongs to $\mathcal{P}'_V$. The intersection $\mathcal{P}'_V\cap\mathcal{P}'_{\max}$ is therefore nonvoid, similarly as in case (i). □

Proposition 2. *Suppose that* $\xi\in\mathfrak{g}^*$ *and* $(\mathfrak{p},\mathfrak{a},\lambda)\in\mathcal{P}'_{\max}$ *with* $\lambda=\xi|_{\mathfrak{p}}$. *If* ξ *vanishes on* $\mathfrak{z}(\xi)$, *then* $\xi(\mathfrak{p}^{[p]})=0$. *In this case the one-dimensional* $\mathfrak{p}$*-module* k_λ *on which* $\mathfrak{p}$ *operates via* λ *has* p*-character* λ, *and so* $U_\xi(\mathfrak{g})\otimes_{U_\lambda(\mathfrak{p})}k_\lambda$ *is an irreducible* $\mathfrak{g}$*-module of dimension* $p^{\frac{1}{2}(\dim\mathfrak{g}-\dim\mathfrak{z}(\xi))}$.

Proof. For each subspace $\mathfrak{h}\subset\mathfrak{g}$ denote by $\mathfrak{h}^\perp\subset\mathfrak{g}$ its orthogonal complement with respect to β_ξ. One has then $(\mathfrak{h}^\perp)^\perp=\mathfrak{h}+\mathfrak{z}(\xi)$. Consider a $(\mathfrak{p},\mathfrak{a},\lambda)$-admissible chain (3). Put $\widetilde{\mathfrak{p}}=\widetilde{\mathfrak{p}}_n$ and $\mathfrak{p}'=\mathfrak{a}'_n$. Note that $\mathfrak{a}'_{i-1}\subset\mathfrak{a}'_i$ for all $i=1,\dots,n$. It follows then from (5) by induction on i that $\widetilde{\mathfrak{p}}_i=\mathfrak{a}_i'^{\perp}$ for each i. For $i=n$ we obtain $\mathfrak{p}'^{\perp}=\widetilde{\mathfrak{p}}$. Hence $\widetilde{\mathfrak{p}}^\perp=\mathfrak{p}'+\mathfrak{z}(\xi)$. Note that $\mathfrak{z}(\xi)\subset\mathfrak{p}$ since $\mathfrak{p}$ is a maximal totally isotropic subspace of $\mathfrak{g}$ with respect to β_ξ. Under the hypotheses of Proposition 2 $\mathfrak{z}(\xi)\subset\mathfrak{p}\cap\ker\xi=\mathfrak{p}'$. Thus $\widetilde{\mathfrak{p}}^\perp=\mathfrak{p}'$.

Observe that $[\widetilde{\mathfrak{p}}, \mathfrak{p}^{[p]}] \subset [\mathfrak{p}, \mathfrak{p}]$ since $\mathfrak{p}$ is an ideal of $\widetilde{\mathfrak{p}}$ by Lemma 2. Hence ξ vanishes on $[\widetilde{\mathfrak{p}}, \mathfrak{p}^{[p]}]$, and so $\mathfrak{p}^{[p]} \subset \widetilde{\mathfrak{p}}^{\perp} = \mathfrak{p}'$. This shows that $\xi(\mathfrak{p}^{[p]}) = 0$. The claim about irreducibility follows from Lemma 1, and the dimension formula follows from the equality $\dim \mathfrak{g} - \dim \mathfrak{p} = \frac{1}{2}\big(\dim \mathfrak{g} - \dim \mathfrak{z}(\xi)\big)$. □

Proposition 3. *Suppose that* $\xi \in \mathfrak{g}^*$ *and* $(\mathfrak{p}, \mathfrak{a}, \lambda) \in \mathcal{P}_{\max}$ *with* $\lambda = \xi|_{\mathfrak{p}}$. *Then every maximal torus of* $\mathfrak{z}(\xi)$ *is a maximal torus of* $\mathfrak{p}$.

Proof. Consider a chain (1) satisfying (2). We have $\mathfrak{a}_i^{\perp} = \mathfrak{p}_i$, and therefore $\mathfrak{p}_i^{\perp} = \mathfrak{z}(\xi) + \mathfrak{a}_i$. As $\mathfrak{a}_i$ is an ideal of $\mathfrak{p}_{i-1}$, we get $[\mathfrak{p}_{i-1}, \mathfrak{a}_i^{[p]}] \subset [\mathfrak{a}_i, \mathfrak{a}_i]$ for $i > 0$, which is contained in the kernel of ξ. This shows that $\mathfrak{a}_i^{[p]} \subset \mathfrak{p}_{i-1}^{\perp} = \mathfrak{z}(\xi) + \mathfrak{a}_{i-1}$.

Denote by $\mathfrak{b}_i$ the $[p]$-closure of $\mathfrak{a}_i$. Then $\mathfrak{b}_i$ is an ideal of $\mathfrak{p}$ since so is $\mathfrak{a}_i$. Hence $\mathfrak{z}(\xi) + \mathfrak{b}_i$ is a p-subalgebra for each i, and it follows that $\mathfrak{b}_i^{[p]} \subset \mathfrak{z}(\xi) + \mathfrak{b}_{i-1}$.

Suppose that $\mathfrak{t}$ is a maximal torus of $\mathfrak{z}(\xi)$ and $s \in \mathfrak{p}$ is a $[p]$-semisimple element which centralizes $\mathfrak{t}$. We will prove that $s \in \mathfrak{t} + \mathfrak{b}_i$ by the downward induction on $i = 0, \ldots, n$. For $i = n$ the assertion is clear since $\mathfrak{t} + \mathfrak{b}_n = \mathfrak{p}$. Suppose that $s \in \mathfrak{t} + \mathfrak{b}_i$ for some $i > 0$. Then $s = t + x$, where $t \in \mathfrak{t}$, $x \in \mathfrak{b}_i$ and $[t, x] = 0$. By the above $s^{[p]} = t^{[p]} + x^{[p]} \in \mathfrak{z}(\xi) + \mathfrak{b}_{i-1}$. Since s is a linear combination of elements $s^{[p^r]}$ with $r > 0$, we get $s \in \mathfrak{z}(\xi) + \mathfrak{b}_{i-1}$. The p-Lie algebra $\mathfrak{h}_i = \big(\mathfrak{z}(\xi) + \mathfrak{b}_{i-1}\big)/\mathfrak{b}_{i-1}$ is a homomorphic image of $\mathfrak{z}(\xi)$, and therefore the image of $\mathfrak{t}$ in $\mathfrak{h}_i$ is a maximal torus of $\mathfrak{h}_i$ by [7, Theorem 2.16]. It follows that $s \in \mathfrak{t} + \mathfrak{b}_{i-1}$, providing the induction step. We can now conclude that $s \in \mathfrak{t} + \mathfrak{b}_0 = \mathfrak{t}$, and the proof is complete. □

Corollary 1. *If* $\mathfrak{z}(\xi)$ *is* $[p]$*-nilpotent, then so too is* $\mathfrak{p}$.

We come to the main result of this section:

Theorem 1. *Let* $\mathfrak{g}$ *be a solvable finite dimensional* p*-Lie algebra over an algebraically closed field of characteristic* $p > 2$, *and let* $\xi \in \mathfrak{g}^*$.

(i) *The algebra* $U_\xi(\mathfrak{g})$ *is simple if and only if* β_ξ *is nondegenerate.*

(ii) *If* β_ξ *is nondegenerate, then* ξ *admits a* $[p]$*-nilpotent polarization* $\mathfrak{p}$ *such that* $\xi(\mathfrak{p}^{[p]}) = 0$, *and the single irreducible* $U_\xi(\mathfrak{g})$*-module is induced from the one-dimensional* $U_\xi(\mathfrak{p})$*-module on which* $\mathfrak{p}$ *operates via* ξ.

Proof. Suppose that β_ξ is nondegenerate so that $\mathfrak{z}(\xi) = 0$. By Proposition 1 there exists $(\mathfrak{p}, \mathfrak{p}, \lambda) \in \mathcal{P}'_{\max}$ such that $\lambda = \xi|_{\mathfrak{p}}$. Then $\mathfrak{p}$ is $[p]$-nilpotent by Corollary 1. By Proposition 2 $U_\xi(\mathfrak{g}) \otimes_{U_\lambda(\mathfrak{p})} k_\lambda$ is an irreducible $\mathfrak{g}$-module of dimension $p^{\frac{1}{2}\dim \mathfrak{g}}$. Since $U_\xi(\mathfrak{g})$ is of dimension $p^{\dim \mathfrak{g}}$, it has to be simple. This proves (ii) and also one implication in (i).

Suppose now that $U_\xi(\mathfrak{g})$ is simple, and let V be its irreducible module. In view of Proposition 1, there exists $(\mathfrak{p}, \mathfrak{p}, \lambda) \in \mathcal{P}'_{\max}$ such that $V_\lambda \neq 0$. Let $0 \neq v \in V_\lambda$ so that $kv \subset V_\lambda$ is a one-dimensional irreducible $U_\xi(\mathfrak{p})$-submodule. By Lemma 1 the $\mathfrak{g}$-module $U_\xi(\mathfrak{g}) \otimes_{U_\xi(\mathfrak{p})} kv$ is irreducible, hence of dimension $p^{\frac{1}{2}\dim \mathfrak{g}}$. Therefore $\dim \mathfrak{p} = \frac{1}{2}\dim \mathfrak{g}$. Let $\eta \in \mathfrak{g}^*$ be any linear function such that $\eta|_{\mathfrak{p}} = \lambda$. By Proposition 1 $\mathfrak{p}$ is a maximal totally isotropic subspace of $\mathfrak{g}$ with respect to β_η. The well-known formula $\dim \mathfrak{g} + \dim \mathfrak{z}(\eta) = 2 \dim \mathfrak{p}$ now yields $\mathfrak{z}(\eta) = 0$. By Proposition 2 applied to the linear function η in place of ξ the p-character of the $\mathfrak{p}$-module kv equals λ. Hence $\lambda = \xi|_{\mathfrak{p}}$. We may thus use $\eta = \xi$ in the argument above to conclude that $\mathfrak{z}(\xi) = 0$. The proof is complete. □

2. Frobenius Lie algebras with exponentiable adjoint derivations

Let $\mathfrak{g}$ be an arbitrary finite dimensional p-Lie algebra over the ground algebraically closed field k. We want to compare two sets

$$\mathcal{X} = \{\xi \in \mathfrak{g}^* \mid U_\xi(\mathfrak{g}) \text{ is simple}\}, \quad \mathcal{Y} = \{\xi \in \mathfrak{g}^* \mid \beta_\xi \text{ is nondegenerate}\}.$$

Lemma 6. *There exists a homogeneous polynomial function f on the vector space $V = \mathfrak{g}^* \oplus k$ such that*

$$\mathcal{X} = \{\xi \in \mathfrak{g}^* \mid f(\xi, 1) \neq 0\}, \quad \mathcal{Y} = \{\xi \in \mathfrak{g}^* \mid f(\xi, 0) \neq 0\}.$$

Proof. Let $n = \dim \mathfrak{g}$. We will exploit the algebraic family of p^n-dimensional associative algebras $U_{\xi,\lambda} = U_{\xi,\lambda}(\mathfrak{g})$ parameterized by points $(\xi, \lambda) \in V$ (see [4]). The algebra $U_{\xi,\lambda}$ contains $\mathfrak{g}$ as a generating subspace and has defining relations

$$xy - yx = \lambda[x, y], \quad x^p = \lambda^{p-1}x^{[p]} + \xi(x)^p \cdot 1 \quad (x, y \in \mathfrak{g}).$$

In particular, two special cases of these algebras are $U_{\xi,1} \cong U_\xi(\mathfrak{g})$ and $U_{\xi,0} \cong S_\xi(\mathfrak{g})$, the factor algebra of the symmetric algebra $S(\mathfrak{g})$ by its ideal generated by all elements $x^p - \xi(x)^p \cdot 1$ with $x \in \mathfrak{g}$.

There is a p-representation $\mathrm{ad}_{\xi,\lambda} : \mathfrak{g} \to \mathrm{Der}\, U_{\xi,\lambda}$ such that $\mathrm{ad}_{\xi,\lambda}(x)(y) = [x, y]$ for $x, y \in \mathfrak{g}$. In this way $U_{\xi,\lambda}$ may be regarded as a module algebra over the restricted universal enveloping algebra $U_0(\mathfrak{g})$ and as a module over the smash product algebra $R_{\xi,\lambda} = U_{\xi,\lambda} \# U_0(\mathfrak{g})$. Let

$$\varphi_{\xi,\lambda} : R_{\xi,\lambda} \to T_{\xi,\lambda} = \mathrm{End}_k\, U_{\xi,\lambda}$$

denote the corresponding representation. Note that $\dim R_{\xi,\lambda} = \dim T_{\xi,\lambda} = p^{2n}$. Hence the map $\varphi_{\xi,\lambda}$ is bijective if and only if $U_{\xi,\lambda}$ is a simple $R_{\xi,\lambda}$-module. Now the $R_{\xi,\lambda}$-submodules of $U_{\xi,\lambda}$ are precisely those left ideals that are stable under the action $\mathrm{ad}_{\xi,\lambda}$. When $\lambda \neq 0$ such left ideals are precisely the two-sided ideals, and the simplicity of $U_{\xi,\lambda}$ as a $R_{\xi,\lambda}$-module is equivalent to the simplicity as an algebra. In particular,

$$\mathcal{X} = \{\xi \in \mathfrak{g}^* \mid \varphi_{\xi,1} \text{ is bijective}\}.$$

On the other hand, according to [4, Proposition 3.4] the algebra $S_\xi(\mathfrak{g})$ has a unique maximal $\mathfrak{g}$-invariant ideal I, and the codimension of this ideal is $p^{\mathrm{codim}_{\mathfrak{g}}\, \mathfrak{z}(\xi)}$. In order that $S_\xi(\mathfrak{g})$ be a simple $R_{\xi,0}$-module, it is necessary and sufficient that $I = 0$, which amounts to $\mathfrak{z}(\xi) = 0$, that is, to $\xi \in \mathcal{Y}$. It follows that

$$\mathcal{Y} = \{\xi \in \mathfrak{g}^* \mid \varphi_{\xi,0} \text{ is bijective}\}.$$

It remains to show that the bijectivity of $\varphi_{\xi,\lambda}$ can be expressed by means of the condition $f(\xi, \lambda) \neq 0$ for a suitable homogeneous polynomial function f on V. We may view $R_{\xi,\lambda}$ and $T_{\xi,\lambda}$ as fibers of two algebraic vector bundles R and T over V. Let $e_1, \ldots, e_n$ be any basis for $\mathfrak{g}$. The monomials $e_1^{a_1} \cdots e_n^{a_n}$ with $0 \leq a_i < p$ form a basis for each $U_{\xi,\lambda}$. These monomials give rise to a basis for each $R_{\xi,\lambda}$ and a basis for each $T_{\xi,\lambda}$, yielding trivializations of R and T. The entries of the matrix of $\varphi_{\xi,\lambda}$ in the above bases are polynomial functions in (ξ, λ). Taking $f(\xi, \lambda)$ to be the determinant of this matrix, we see that $\varphi_{\xi,\lambda}$ is bijective if and only if $f(\xi, \lambda) \neq 0$.

As explained in [4], for each $0 \neq t \in k$ there is a $\mathfrak{g}$-equivariant algebra isomorphism $\theta_t : U_{\xi,\lambda} \to U_{t\xi,t\lambda}(\mathfrak{g})$. Hence the algebra $U_{\xi,\lambda}$ has no nontrivial $\mathfrak{g}$-invariant ideals if and only if so does $U_{t\xi,t\lambda}(\mathfrak{g})$. In other words, bijectivity of $\varphi_{\xi,\lambda}$ is equivalent to bijectivity of $\varphi_{t\xi,t\lambda}$. It follows that the zero locus of the polynomial function f is a conical subset of V, whence f is homogeneous. □

Remark. It is possible to compute the degree of the polynomial function f in Lemma 6 proceeding as follows. The isomorphisms θ_t induce actions of the one-dimensional torus $\mathbb{G}_m$ on R and T. Taking quotients modulo these actions we pass to a morphism of vector bundles $\overline{R} \to \overline{T}$ over the projective space $\mathbb{P}(V)$ associated with V. Let also $\overline{U} = U/\mathbb{G}_m$, where U is the vector bundle over $V \setminus \{0\}$ with fibers $U_{\xi,\lambda}$. Each line bundle over $\mathbb{P}(V)$ is isomorphic to some $L(s)$, defined as the quotient of $(V \setminus \{0\}) \times k$ by the action of $\mathbb{G}_m$ such that $t \cdot (v, c) = (tv, t^s c)$, where $s \in \mathbb{Z}$. The scalar multiples of any monomial $e_1^{a_1} \cdots e_n^{a_n}$ produce a $\mathbb{G}_m$-stable line subbundle of U. This leads to a decomposition

$$\overline{U} \cong \bigoplus_{\{(a_1,\dots,a_n) | 0 \le a_i < p\}} L(-a_1 - \cdots - a_n).$$

The bundle $\overline{R}$ is isomorphic to a direct sum of p^n copies of $\overline{U}$, while $\overline{T} \cong \overline{U} \otimes \overline{U}^*$. As a result, $\bigwedge^{p^{2n}} \overline{R} \cong L(-d)$, where

$$d = p^n \cdot \sum_{\{(a_1,\dots,a_n) | 0 \le a_i < p\}} (a_1 + \cdots + a_n) = \frac{np^{2n}(p-1)}{2},$$

while $\bigwedge^{p^{2n}} \overline{T} \cong L(0)$ is trivial. Now f can be identified with a section of the line bundle $\operatorname{Hom}\bigl(L(-d), L(0)\bigr) \cong L(d)$. This means that $\deg f = d$.

Corollary 2. *If $\mathfrak{g}$ is Frobenius, that is, $\mathcal{Y} \ne \varnothing$, then $f \ne 0$, and therefore $\mathcal{X} \ne \varnothing$.*

Whether $\mathcal{X} \ne \varnothing$ implies $\mathcal{Y} \ne \varnothing$ is a special case of the still open Kac–Weisfeiler conjecture from [8].

Proposition 4. *If $\mathfrak{g}$ is Frobenius and $\mathcal{Y} \subset \mathcal{X}$, then $\mathcal{X} = \mathcal{Y}$.*

Proof. By Lemma 6 the complements $\mathcal{X}^c = \mathfrak{g}^* \setminus \mathcal{X}$ and $\mathcal{Y}^c = \mathfrak{g}^* \setminus \mathcal{Y}$ are hypersurfaces in $\mathfrak{g}^*$. The inclusion $\mathcal{Y} \subset \mathcal{X}$ entails $\mathcal{X}^c \subset \mathcal{Y}^c$. Therefore each irreducible component of $\mathcal{X}^c$ is an irreducible component of $\mathcal{Y}^c$. Since $\mathcal{Y}^c$ is a conical subset of $\mathfrak{g}^*$, so too is each irreducible component of $\mathcal{Y}^c$. It follows that $\mathcal{X}^c$ is a conical subset as well. Hence the polynomial function $\xi \mapsto f(\xi, 1)$ defining $\mathcal{X}^c$ is homogeneous. We can write

$$f(\xi, \lambda) = \sum_{i=0}^{d} f_i(\xi)\lambda^i,$$

where each f_i is a homogeneous polynomial function of degree $d - i$ on $\mathfrak{g}^*$. Since $\mathfrak{g}$ is Frobenius, we have $\mathcal{Y} \ne \varnothing$, whence $f_0 \ne 0$. But then we must have $f_i = 0$ for all $i > 0$, that is, $f(\xi, \lambda)$ does not depend on λ. □

Theorem 2. *Let $\mathfrak{g}$ be a Frobenius p-Lie algebra with the automorphism group G. Suppose that* $\operatorname{ad} \mathfrak{g} \subset \operatorname{Lie} G$. *Then $\mathcal{X} = \mathcal{Y}$.*

Proof. Both $\mathcal{X}$ and $\mathcal{Y}$ are stable under the coadjoint action of G. For any $\xi \in \mathcal{Y}$ the nondegeneracy of β_ξ yields $\mathfrak{g} \cdot \xi = \mathfrak{g}^*$. Hence the tangent space at ξ to the G-orbit $G\xi$ coincides with $\mathfrak{g}^*$, and therefore $G\xi$ is open in $\mathfrak{g}^*$. Since any two nonempty open subsets of $\mathfrak{g}^*$ have nonempty intersection, we conclude that $\mathcal{Y}$ is a single G-orbit. As $\mathcal{X}$ is also nonempty and open in $\mathfrak{g}^*$, we get $\mathcal{X} \bigcap \mathcal{Y} \ne \varnothing$, whence $\mathcal{Y} \subset \mathcal{X}$. Now Proposition 4 applies. □

3. The semisimple locus: an example

Let us now look at a different pair of subsets of $\mathfrak{g}^*$:

$$\mathcal{X} = \{\xi \in \mathfrak{g}^* \mid U_\xi(\mathfrak{g}) \text{ is semisimple}\}, \quad \mathcal{Y} = \{\xi \in \mathfrak{g}^* \mid \mathfrak{z}(\xi) \text{ is toral}\}.$$

It was proved in [4, Section 4] that both of them are open in $\mathfrak{g}^*$ and that $\mathcal{Y} \ne \varnothing$ implies $\mathcal{X} \ne \varnothing$. Moreover, the stabilizers $\mathfrak{z}(\xi)$ of all linear functions $\xi \in \mathcal{Y}$ have equal dimensions. If s denotes their common dimension, then for each $\xi \in \mathcal{X}$ the semisimple algebra $U_\xi(\mathfrak{g})$ has precisely p^s nonisomorphic simple modules, all of equal dimension.

One may ask what are those p-Lie algebras for which $\mathcal{X} = \mathcal{Y}$. For instance, if $\mathfrak{g}$ is the Lie algebra of a simply connected semisimple algebraic group G and p is good for the root system of G, then $\mathcal{X}$ consists precisely of the regular semisimple linear functions [9, Corollary 3.6] so that the equality $\mathcal{X} = \mathcal{Y}$ does hold. In this section, we provide examples of nilpotent p-Lie algebras for which $\mathcal{X} \ne \mathcal{Y}$.

Consider a p-Lie algebra $\mathfrak{g}$ whose center $\mathfrak{t}$ is a toral subalgebra of codimension 2 in $\mathfrak{g}$ and $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{t}$. Let $u, v \in \mathfrak{g}$ span a subspace complementary to $\mathfrak{t}$ in $\mathfrak{g}$. There is an element $0 \ne t \in \mathfrak{t}$ such that $[u,v] = t$. Then $[\mathfrak{g},\mathfrak{g}] = kt$ is a one-dimensional subspace.

Since $\mathfrak{g}$ is nilpotent, it has a largest toral subalgebra. Clearly this subalgebra coincides with $\mathfrak{t}$. Now $\mathfrak{t} \subset \mathfrak{z}(\xi)$ for all $\xi \in \mathfrak{g}^*$. Hence $\mathfrak{z}(\xi)$ is toral if and only if $\mathfrak{z}(\xi) = \mathfrak{t}$. If $\mathfrak{z}(\xi) \ne \mathfrak{t}$, then $\mathfrak{z}(\xi) = \mathfrak{g}$, which occurs precisely when ξ vanishes on $[\mathfrak{g},\mathfrak{g}]$. It follows that

$$\mathcal{Y} = \{\xi \in \mathfrak{g}^* \mid \xi(t) \ne 0\}.$$

Denote by $\mathfrak{t}^{*(1)}$ the vector space of all p-semilinear maps $\mathfrak{t} \to k$, that is, $\mathfrak{t}^{*(1)}$ is the Frobenius twist of the dual space $\mathfrak{t}^*$. The map $\wp : \mathfrak{t}^* \to \mathfrak{t}^{*(1)}$ defined by the rule

$$\wp(\lambda)(x) = \lambda(x)^p - \lambda(x^{[p]}) \quad \text{for } \lambda \in \mathfrak{t}^* \text{ and } x \in \mathfrak{t}$$

is a finite surjective morphism of algebraic varieties. There is also a bijective morphism $\mathfrak{t}^* \to \mathfrak{t}^{*(1)}$ given by $\lambda \mapsto \lambda^p$, where $\lambda^p(x) = \lambda(x)^p$.

With any simple $\mathfrak{g}$-module V one can associate a linear function $\lambda \in \mathfrak{t}^*$ such that each element $x \in \mathfrak{t}$ acts in V as a scalar multiplication by $\lambda(x)$. If ξ is the p-character of V, then $\wp(\lambda) = \xi^p|_{\mathfrak{t}}$. Conversely, for any pair $\lambda \in \mathfrak{t}^*$ and $\xi \in \mathfrak{g}^*$ satisfying the previous equality there is precisely one simple $U_\xi(\mathfrak{g})$-module V which has λ as the associated function. If $\lambda(t) = 0$, then $[\mathfrak{g},\mathfrak{g}]$ annihilates V, whence $\dim V = 1$. Otherwise V is induced from a one-dimensional representation of any abelian subalgebra of codimension 1 in $\mathfrak{g}$ so that $\dim V = p$. Since all fibers of the map $\wp$ have cardinality $N = p^{\dim \mathfrak{t}}$, for each $\xi \in \mathfrak{g}^*$ there are precisely N nonisomorphic simple $U_\xi(\mathfrak{g})$-modules. In order that $U_\xi(\mathfrak{g})$ be semisimple, it is necessary and sufficient that its dimension $p^{\dim \mathfrak{g}}$ be equal to $\sum(\dim V)^2$, the sum over all those modules. This happens precisely when all simple $U_\xi(\mathfrak{g})$-modules have dimension p. We conclude that

$$\mathcal{X} = \{\xi \in \mathfrak{g}^* \mid \lambda(t) \ne 0 \text{ for each } \lambda \in \wp^{-1}(\xi^p|_{\mathfrak{t}})\}.$$

Suppose now that t is such that $t^{[p]} \notin kt$. Then neither $\mathcal{X} \subset \mathcal{Y}$ nor $\mathcal{Y} \subset \mathcal{X}$. To see this let λ and ξ be as above. If $\lambda(t) = 0$, but $\lambda(t^{[p]}) \ne 0$, then the equality $\lambda(t)^p - \lambda(t^{[p]}) = \xi(t)^p$ yields $\xi(t) \ne 0$. In this case $\xi \in \mathcal{Y}$, but $\xi \notin \mathcal{X}$. Now the subspace

$$S = \{\lambda \in \mathfrak{t}^* \mid \lambda(t) = \lambda(t^{[p]}) = 0\}$$

has codimension 2 in $\mathfrak{t}^*$. Hence $\wp(S)$ is a closed subvariety of codimension 2 in $\mathfrak{t}^{*(1)}$, and it follows that there exists $\xi \in \mathfrak{g}^*$ such that $\xi(t) = 0$, but $\xi^p|_{\mathfrak{t}} \notin \wp(S)$. In this case $\xi \notin \mathcal{Y}$, but $\xi \in \mathcal{X}$.

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Резюме

С.М. Скрябин. О локусе p-характеров, определяющих простые редуцированные обертывающие алгебры.

В двух случаях подтверждена гипотеза, утверждающая, что редуцированная обёртывающая алгебра $U_\xi(\mathfrak{g})$ ограниченной алгебры Ли $\mathfrak{g}$ является простой тогда и только тогда, когда альтернирующая билинейная форма, ассоциированная с заданным p-характером $\xi \in \mathfrak{g}^*$, невырождена.

Ключевые слова: ограниченные алгебры Ли, разрешимые алгебры Ли, фробениусовы алгебры Ли, редуцированные обертывающие алгебры.

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