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On the Lattice of Overcommutative Varieties of Monoids

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Abstract—We study the lattice of varieties of monoids, i.e., algebras with two operations, namely, an associative binary operation and a 0-ary operation that fixes the neutral element. It was unknown so far, whether this lattice satisfies some non-trivial identity. The objective of this paper is to give the negative answer to this question. Namely, we prove that any finite lattice is a homomorphic image of some sublattice of the lattice of overcommutative varieties of monoids (i.e., varieties that contain the variety of all commutative monoids). This implies that the lattice of overcommutative varieties of monoids, and therefore, the lattice of all varieties of monoids does not satisfy any non-trivial identity.

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In this paper we study the lattice of varieties of monoids, i.e., the algebras whose signature consists of an associative binary operation and a 0-ary operation separating the neutral element. A large number of works have been devoted to the varieties of monoids, but the vast majority of them is devoted to the monoid identities. As for the lattice of varieties of monoids, that we denote by **MON**, we know very little about it today. Papers [1] and [2] describe lattices of all commutative and all idempotent varieties of monoids, respectively. Paper [3] (theorem 1) gives an example of a variety of monoids without coverings in the lattice **MON**. Paper [4] (section 3.2) presents two monoid varieties such that the subvariety lattice of each of them is finite, and the subvariety lattice of their union is continual and does not meet the maximality condition. Note that certain papers occasionally contain the auxiliary results on the description of the easily organized lattices of subvarieties of certain concrete varieties of monoids (e.g., [5], lemma 4.1). So we obtain a complete list of the currently known results on the lattice **MON**. This contrasts sharply with the large number of bright and deep results obtained to date on the lattice of semigroup varieties (see the review [6]).

Due to the poor knowledge on the lattice of varieties of monoids, many natural questions related to it remain open to this day. In particular, up to the present time it has not been known whether this lattice meets any non-trivial identity. The main purpose of this paper is to give a negative answer to this question. We note that for a lattice of varieties of semigroups a negative answer to a similar question was obtained in 1971 in the works of C. Barris and E. Nelson [7, 8].

A variety of monoids is called *overcommutative* if it contains the variety of all commutative monoids. It is clear that the collection of all overcommutative varieties of monoids forms a sublattice in the lattice of all varieties of monoids. The main result of this paper is the following

Theorem. *The lattice of overcommutative varieties of monoids does not meet any nontrivial identity.*

Note that an analog of this theorem for varieties of semigroups follows from [9], corollary 4.4.

Also note that absence of nontrivial identities in the lattice of **MON** was recently independently proved by I. A. Mikhailova. Namely, it states that a lattice anti-isomorphic to the partition lattice of a countable set is embedded in the lattice of periodic varieties of monoids (a result of oral communication). It is well-known that the fulfillment of the latter property in the lattice implies absence of nontrivial identities in it (e.g., [10], corollary IV.4.6). It seems pertinent to note here that the lattice of overcommutative varieties

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