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Implicit Algebraic Geometry on Categories of Universal Algebras

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Abstract—Using the notion of an implicit operation on universal algebras, we redefine basic notions of the algebraic geometry of universal algebras. The results obtained for the implicit algebraic geometry imply (as special cases) the known results on the conditional geometric and algebraic geometric comparability of algebras.

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The basic notions and fundamentals of the theory of algebraic geometry for arbitrary varieties of universal algebras based on the notion of an equation for such varieties were developed by B. I. Plotkin and V. I. Remeslennikov et al. (e.g., [1–3]). The equation is understood as the equality of two terms of the variety under consideration. An important role in this theory is played by relations $\overset{\Delta}{\sim}$ of the geometric equivalence and the relation $\leq^{\Delta}$ of the geometric comparability of algebras introduced in [4–5].

In the paper [6] one makes an attempt to extend the basic notions and constructions of the algebraic geometry of universal algebras to the case when terms are replaced with conditional terms (see details, e.g., in [7], P. 88; [8]) and, respectively, equations are replaced with conditional equations.

In this paper we make an analogous attempt to replace terms (conditional terms) with implicit operations over categories of universal algebras. As a result, we obtain the known properties of the algebraic and conditional algebraic geometries as special cases of results on the implicit algebraic geometry with a proper choice of categories of universal algebras.

Let $\mathcal{K}$ be a class of universal algebras of some fixed signature σ. Assume that $\overrightarrow{\mathcal{K}}$ is a category whose objects are $\mathcal{K}$-algebras, while morphisms are certain homomorphisms (some or all of them) of certain $\mathcal{K}$-algebras to other ones. Recall (e.g., [9]) that an implicit n-argument operation f on the category $\overrightarrow{\mathcal{K}}$ is some system of n-argument functions $f_{\mathcal{A}}$ (for each algebra $\mathcal{A} = \langle A; \sigma\rangle$ from $\mathcal{K}$) defined on the basic sets of $\mathcal{K}$-algebras such that they commute with $\overrightarrow{\mathcal{K}}$-morphisms, i.e., such that for algebras $\mathcal{A}$ and $\mathcal{B}$ from $\mathcal{K}$ for any $\varphi \in \mathrm{Hom}_{\overrightarrow{\mathcal{K}}}(\mathcal{A}, \mathcal{B})$ and any $a_1, \ldots, a_n \in \mathcal{A}$ it holds

$$\varphi(f_{\mathcal{A}}(a_1, \ldots, a_n)) = f_{\mathcal{B}}(\varphi(a_1), \ldots, \varphi(a_n)),$$

and such that for any $\mathcal{A} \in \mathcal{K}$ and any $a_1, \ldots, a_n \in \mathcal{A}$,

$$f_{\mathcal{A}}(a_1, \ldots, a_n) \in \langle a_1, \ldots, a_n\rangle_{\mathcal{A}},$$

where $\langle a_1, \ldots, a_n\rangle_{\mathcal{A}}$ is a subalgebra of the algebra $\mathcal{A}$ generated by the set $\{a_1, \ldots, a_n\}$.

Since morphisms from $\mathrm{Hom}_{\overrightarrow{\mathcal{K}}}(\mathcal{A}, \mathcal{B})$ are some homomorphisms between $\mathcal{K}$-algebras, any termal functions on $\mathcal{K}$-algebras are implicit operations on the category $\overrightarrow{\mathcal{K}}$. Note also that superpositions of implicit on $\overrightarrow{\mathcal{K}}$ operations are also implicit operations on the category $\overrightarrow{\mathcal{K}}$. Let the symbol $IF_n(\overrightarrow{\mathcal{K}})$ denote the totality of n-argument implicit operations on the category $\overrightarrow{\mathcal{K}}$.

Note that in the case, when $\mathcal{K}$ is a variety of algebras, while $\overrightarrow{\mathcal{K}}$-morphisms are all homomorphisms of some $\mathcal{K}$-algebras to other ones, $\overrightarrow{\mathcal{K}}$-implicit operations on $\mathcal{K}$-algebras are termal. In the case, when $\mathcal{K}$

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