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ISOMORPHISM OF SPACES OF ULTRADIFFERENTIABLE FUNCTIONS OF THE BEURLING TYPE

A.V. Abanin

1. Problem definition and the main result. In [1], [2], certain modification of the well known Beurling–Björck approach (see [3], [4]) to the definition and the study of non-quasianalytic classes of functions is proposed. Namely, continuous non-decreasing on $[0,\infty)$ functions $\omega : [0,\infty) \to [0,\infty)$ which satisfy the following restrictions:

$$(\alpha)\ \ \omega(2t) \le K(\omega(t)+1) \ \text{ on } \ [0,\infty), \qquad (\beta)\ \ \int_1^\infty \frac{\omega(t)}{t^2}dt < \infty,$$

$$(\gamma)\ \ \ln t = o(\omega(t)) \ \text{ for } \ t\to\infty, \qquad (\delta)\ \ \varphi_\omega(x) = \omega(e^x) \ \text{ is convex on } \ [0,\infty)$$

are considered. These functions ω are called the *weight functions* or simply *weights*. We associate each weight ω with the space

$$\mathcal{E}_{(\omega)}(\mathbb{R}^N) := \{ f \in C^\infty(\mathbb{R}^N) \,|\, \forall p \in \mathbb{N} \ \ \|f\|_p^{(\omega)} := \sup_{\alpha\in\mathbb{N}_0^N} \sup_{|x|\le p} |f^{(\alpha)}(x)|\, e^{-p\varphi_\omega^*(|\alpha|/p)} < \infty \}$$

with a topology defined by the set of pre-norms $(\|\cdot\|_p^{(\omega)})_{p\in\mathbb{N}}$. Here $f^{(\alpha)}$ is a derivative of f corresponding to the multiindex $\alpha = (\alpha_1,\dots,\alpha_N)$; $|\alpha| := \alpha_1+\dots+\alpha_N$; $|x| := |x_1|+\dots+|x_N|$ for $x=(x_1,\dots,x_N)\in\mathbb{R}^N$; $\varphi_\omega^*(s) := \sup\{ts - \varphi_\omega(t) : t\ge 0\}$ is the Young function conjugate to φ_ω. The space $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ is called that of ω-ultradifferentiable functions of the Beurling type; it is the nuclear Frechet space ([2], proposition 4.9).

We say that two weights ω and σ are *equivalent* ($\omega\sim\sigma$) if simultaneously $\omega(t) = O(\sigma(t))$ and $\sigma(t) = O(\omega(t))$ for $t\to\infty$. In other words, $\omega\sim\sigma$ if there exists $C\ge 1$ such that

$$\frac{1}{C}\,\omega(t) - C \le \sigma(t) \le C\omega(t) + C \quad \forall t\in[0,\infty). \tag{1}$$

The fulfillment of (1) for all $y\ge 0$ implies

$$C\varphi_\omega^*\left(\frac{y}{C}\right) - C \le \varphi_\sigma^*(y) \le \frac{1}{C}\,\varphi_\omega^*(Cy) + C.$$

Therefore, as one can easily see, for equivalent weights ω and σ, the spaces $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{E}_{(\sigma)}(\mathbb{R}^N)$ coincide as sets and topologically (however, here the first case implies the second one). Theorem 1.3.18 of [4] implies the following assertion. If ω and σ are upper half-additive weight functions, then the inverse implication is also valid, i. e., the equality $\mathcal{E}_{(\omega)}(\mathbb{R}^N) = \mathcal{E}_{(\sigma)}(\mathbb{R}^N)$ implies the equivalence of ω and σ. Using the proposed in [4] method which is based on the Banach theorem about an open mapping, U. Franken proved the same fact for arbitrary weight functions (this result was not published, but professor R. Meise informed the author about it). In this paper, we prove the stronger assertion.

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