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Solution of Fifth-Degree Equations

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Abstract—In this paper we establish a relationship between two approaches to the solution of algebraic fifth-degree equations, namely, the Hermite–Kronecker method (based on the modular elliptic equation) and the Mellin method (based on hypergeometric series).

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1. THE GENERAL SCHEME FOR SOLVING FIFTH-DEGREE EQUATIONS

In 1789 the Swedish mathematician E. S. Bring [1] with the help of the Tschirnhaus transform reduced the general algebraic equation of the fifth degree to the form

$$y^5 + 5y = a \tag{1}$$

(see also [2, 3]). It is well-known ([2, 4]) that Eq. (1) is solvable in terms of modular equations. The following theorem is valid.

Theorem (Hermite–Kronecker). *Solutions to the equation*

$$y^5 + 5y = a$$

obey the formula

$$y_l(a) = \frac{a}{\omega_l^2(a) + 5}, \quad l = 0, 1, 2, 3, 4, \tag{2}$$

where

$$\omega_l(a) = \frac{(v_\infty - v_l)(v_{l+1} - v_{l-1})(v_{l+2} - v_{l-2})}{\sqrt{5}\, f^3(\tau)} \tag{3}$$

(*indices at* v *are taken modulo* 5)*,* f *and* a *are connected by the correlation*

$$f^{12}(\tau) = \frac{a^2 \pm \sqrt{a^4 + 256}}{2}. \tag{4}$$

Here $v_0, \dots, v_4, v_\infty$ are certain extensions and shifts of the modular function

$$u = f(\tau) = q^{-\frac{1}{24}} \prod_{k=1}^{\infty} (1 + q^{2k-1}) \Big|_{q=e^{i\pi\tau}},$$

namely,

$$v_0 = f\left(\frac{\tau}{5}\right), \quad v_1 = f\left(\frac{\tau + 96}{5}\right), \quad v_2 = f\left(\frac{\tau - 48}{5}\right), \quad v_3 = f\left(\frac{\tau + 48}{5}\right),$$

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