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A NEW CRITERION FOR AN OPERATOR TO BELONG TO THE CLASS $K(H)$

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Among the central problems of the theory of operators acting in spaces with indefinite metric there stands the problem of existence of maximal semidefinite subspaces which are invariant with respect to operators acting in these spaces.

T.Ya. Azizov in [1] introduced and described a class $K(H)$ which is composed by operators acting in spaces with indefinite metric and possessing the following properties: $\Phi \in K(H)$ if it has at least one pair N_+ (of maximal non-negative), N_- (maximal non-positive) invariant subspaces such that $N_+[\perp]N_-$ and $N_\pm$ admit the decompositions

$$N_+ = N^0[+]N^{++}, \quad N_- = N^0[+]N^{--}, \tag{1}$$

where $\dim N^0 < \infty$, N^0 is the isotropic subspace, N^{++} is the uniformly positive subspace, and N^{--} is the uniformly negative subspace. Here and on we use a conventional "indefinite" terminology and notation (see, e.g., [1]).

Let H be the Kreĭn space with indefinite metric $[x,y] = (Jx,y)$, where J is the canonical symmetry of the space H. The closed linear span (l. s.) of root vectors of the operator Φ is denoted by $E(\Phi)$, while the closed l. s. of its eigenvectors — by $E_0(\Phi)$.

Theorem. *Let Φ be a J-selfadjoint operator in the Kreĭn space H, where the Riesz basis $\{f_i\}$ exists composed by root vectors of the operator Φ. In addition, $\dim(E(\Phi)/E_0(\Phi)) < \infty$ and the non-real spectrum of the operator Φ consists of no more than a finite number of eigenvalues with regard for their multiplicities. Then Φ is an operator of class $K(H)$.*

Proof. Let us show that, in possessing all above-cited properties, the operator Φ will be an operator of class $K(H)$. To this end we shall indicate the necessary pair of subspaces N_+, N_-.

Let $\mu_1, \bar\mu_1, \mu_2, \bar\mu_2, \dots, \mu_m, \bar\mu_m$ be all different non-real eigenvalues of the operator Φ. Consider the subspaces: $N^1 = \mathrm{L.S.}\,\{N(\mu_i)\}_{i=1,2,\dots,m}$, $N^2 = \mathrm{L.S.}\,\{N(\bar\mu_i)\}_{i=1,2,\dots,m}$, where $N(\mu_i)$ and $N(\bar\mu_i)$ are root spaces of the operator Φ, which correspond to μ_i and $\bar\mu_i$, respectively, $i = 1,2,\dots,m$. By virtue of [1] (see Chap. II, § 3, p. 132) the subspaces N^1 and N^2 are neutral.

From the normality of the eigenvalues μ_i and $\bar\mu_i$ of the operator Φ ($i = 1,2,\dots,m$) (ibid, Chap. II, § 2, p. 124; Chap. I, § 7, p. 63) it follows that the subspace $H_1 = \mathrm{L.S.}\,\{N^1, N^2\}$, which is invariant with respect to the operator Φ, is nondegenerate and projectively complete. Therefore, $H = H_1[+]H_2$, where $H_2 = H_1^{[\perp]}$ is the Kreĭn space, invariant with respect to the operator Φ (ibid, Chap. I, § 7, pp. 62, 64).

Note that the operator $\Phi \mid H_2$ inherits the above-mentioned properties of the operator Φ, i. e., non-real spectrum of the operator $\Phi \mid H_2$ consists of a finite (empty) set of non-real eigenvalues.

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