ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2007, Vol. 51, No. 6, pp. 52–60. © Allerton Press, Inc., 2007.
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Several Stability Tests for Linear Autonomous Differential Equations with Distributed Delay

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Received March 21, 2006

DOI: 10.3103/S1066369X07060072

In this paper, we study the stability of a solution to the *distributed delay* equation

$$\begin{aligned} &\dot{x}(t) + ax(t) + b\int_{t-\tau-h}^{t-\tau} x(s)ds = f(t), \quad t \in \mathbb{R}_+; \\ &x(\xi) = 0, \quad \xi < 0, \end{aligned} \tag{1}$$

where the parameters a, b, τ, and h are assumed to be constant.

The stability of linear functional differential equations is considered in many papers. Note that most of them deal with equations with a concentrated delay. For these equations the efficient stability tests are proposed (see [1–7] and references therein). The amount of stability tests for equations in form (1) is much less. In paper [8], an asymptotic stability criterion is obtained for a solution to Eq. (1) with $a = 0$ and real b (evidently, for the first time). In papers [9–12] the sufficient stability conditions are formulated for Eqs. (1) with variable coefficients a, b, and $\tau = 0$. Inserting the constant coefficients into their theorems, one obtains the tests which are far from the exact ones.

The objective of this paper is to formulate the asymptotic stability criteria for solutions to Eq. (1) in terms of the coefficients of the initial problem.

Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$; let $\mathbb{C}$ be the space of complex numbers.

Definition 1. We understand a *solution to Eq.* (1) as a function $x : \mathbb{R}_+ \to \mathbb{C}$ which is absolutely continuous on any finite segment and satisfies (1) almost everywhere.

It is well known ([1], P. 84, theorem 1.1) that Eq. (1) is uniquely solvable and its solution admits the representation

$$x(t) = C(t, 0)x(0) + \int_0^t C(t, s)f(s)ds, \tag{2}$$

where the function $C(t, s)$ is called the *Cauchy function* for Eq. (1) ([1], P. 84).

Consider the function $x_0 : \mathbb{R}_+ \to \mathbb{C}$ which is a solution to the problem

$$\begin{aligned} &\dot{x}_0(t) + ax_0(t) + b\int_{t-\tau-h}^{t-\tau} x_0(s)ds = 0, \quad t \in \mathbb{R}_+; \\ &x_0(0) = 1, \quad x_0(\xi) = 0, \quad \xi < 0. \end{aligned} \tag{3}$$

We call the mentioned function a *fundamental solution* ([1], P. 34).

Note the following fact: since a, b, τ, and h are independent of t, the functions $x_0(t)$ and $C(t, s)$ are connected by the relation

$$x_0(t - s) = C(t, s). \tag{4}$$

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