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Semilinear Metric Semilattices on $\mathbb{R}$-trees

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1. INTRODUCTION

The notion of an $\mathbb{R}$-tree is a generalization of the notion of a simplicial tree and is included into a more general family of so-called Λ-trees. A geodesic metric space X is called an $\mathbb{R}$-tree if, in any of its triangles, each of the sides is contained in the union of the two other sides. In this paper, we introduce the notion of a metric semilattice on a metric space and prove the following criterion.

Theorem 1. *Let X be a geodesic space. If X is an $\mathbb{R}$-tree, then, for any point $o \in X$, there exists a unique partial order on X with respect to which this space is an upper semilinear metric $\vee$-semilattice with root o. With respect to such an order, each nonempty subset $A \subset X$ has the least upper bound. Conversely, if X admits a partial order which makes X into a semilinear metric semilattice with the common direction of the semilinearity and the semilattice, then X is an $\mathbb{R}$-tree.*

We study the set $\mathcal{O}_+(X)$ of partial orders on a complete locally compact $\mathbb{R}$-tree X which define on X upper semilinear metric $\vee$-semilattices. We introduce a topology on $\mathcal{O}_+(X)$. On the subspace $\mathcal{O}^r_+(X) \subset \mathcal{O}_+(X)$ consisting of root orders this topology is generated by the Hausdorff metric on the family $\mathcal{C}(X \times X)$ of closed subsets of the metric square $X \times X$. An extension of the topology to all $\mathcal{O}_+(X)$ is constructed with the use of the base of neighborhoods of nonroot orders. We prove the following theorem.

Theorem 2. *The metric space $\mathcal{O}^r_+(X)$ is isometric to X, and the topological space $\mathcal{O}_+(X)$ is homeomorphic to the metric compactification $\overline{X}_m$ of X.*

As an application of Theorem 1, in Section 5, we construct an example which demonstrates the importance of the condition of local compactness in the following conjecture formulated in [1].

Conjecture. *Every locally compact similarly homogeneous nonhomogeneous metric space with intrinsic metric (X, ρ) is homeomorphic to the topological product $F \times \mathbb{R}_+$, where F is an arbitrary level set of the completeness radius function on X. The topological group $\mathrm{Sim}(X)$ of similarities of X is homeomorphic to the direct topological product $\mathrm{Isom}(X) \times \mathbb{R}_+$, where $\mathrm{Isom}(X) \subset \mathrm{Sim}(X)$ is the subgroup of isometries of X.*

$\mathbb{R}_+$ is the set of positive real numbers. The metric space X we have constructed is an $\mathbb{R}$-tree and satisfies all conditions of the conjecture except for the condition of local compactness. It is not homeomorphic to the topological product $F \times \mathbb{R}_+$, but it is a metric fibration in the sense of the definition given in [2]. Each point the space X is its branch point. The group of similarities $\mathrm{Sim}(X)$ splits with the use of the exact sequence

$$0 \to \mathrm{Isom}(X) \to \mathrm{Sim}(X) \to \mathbb{R}_+ \to 0, \tag{1}$$

but is not homeomorphic to the topological group $\mathrm{Isom}(X) \times \mathbb{R}_+$.

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