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Fully Idempotent Homomorphisms

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Abstract—For arbitrary modules A and B we introduce and study the notion of a fully idempotent $\mathrm{Hom}\,(A,B)$. As a corollary we obtain some well-known properties of fully idempotent rings and modules.

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1. INTRODUCTION

In this paper we denote by R a ring with a unit element. All considered modules are right and unitary. If A and B are right R-modules, then we denote by $\mathrm{Hom}(A,B)$ the set of all R-homomorphisms from A to B.

A ring R is said to be *fully idempotent from the right* (*from the left*) if $I^2 = I$ for each right (left) ideal I of the ring R. If the equality $I^2 = I$ is fulfilled for each ideal I of a ring R, then R is said to be *fully idempotent*. It is not difficult to see that for a ring R the following conditions are equivalent:

1) R is fully idempotent from the right,

2) for each element r of the ring R there exist elements $r_1,\dots,r_n,s_1,\dots,s_n \in R$ such that $r = \sum_{i=1}^{n} r r_i r s_i$.

Fully idempotent rings are studied in [1–3] and other papers. Some properties of fully idempotent rings are described in monographs [4, 5] and in the review [6].

Let M be an arbitrary right R-module and $S = \mathrm{End}_R(M)$. Denote by M^* the left R-module of $\mathrm{Hom}_R(M,R_R)$. Let the symbol $[m,f]$ stand for an endomorphism of the module M such that $[m,f](n) = mf(n)$ for each $n \in M$. A right R-module M is said to be *fully idempotent from the right* (*from the left*) if $m \in [m,M^*]mR$ ($m \in \mathrm{End}_R(M)[m,M^*]m$) for each $m \in M$. If $m \in \mathrm{End}_R(M)[m,M^*]mR$ for each $m \in M$, then the module M is said to be *fully idempotent*. Fully idempotent modules are considered in papers [7–11].

A right R-module M is said to be *regular in the sense of Zelmanowitz* [12] if for each $m \in M$ there exists a homomorphism $f: M \to R$ such that $m = mf(m)$. Modules regular in the sense of Zelmanowitz are studied in [7, 8, 12, 13].

Let A and B be right R-modules. A homomorphism $f \in \mathrm{Hom}(A,B)$ is said to be *regular* if there exists a homomorphism $g \in \mathrm{Hom}(B,A)$ such that $f = fgf$. The set $\mathrm{Hom}(A,B)$ is said to be *regular* if each element of $\mathrm{Hom}(A,B)$ is regular. Regular morphisms are studied in [14–16]. Note that a right R-module M is regular in the sense of Zelmanowitz if and only if the canonically isomorphic to it module $\mathrm{Hom}(R,M)$ is regular.

Let $f \in \mathrm{Hom}(A,B)$. Introduce the following denotations:

$$H_r(f) := \Big\{ \sum_i g_i f s_i \mid g_i \in \mathrm{Hom}(B,A),\ s_i \in \mathrm{Hom}(A,A) \ \text{ for each } i \Big\},$$

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