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ESTIMATES OF THE FOURIER–HAAR COEFFICIENTS FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

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This article is devoted to investigation of the behavior of the Fourier–Haar coefficients $a_{mn}(f)$ of the functions f, which are Lebesgue-integrable on the square $D=[0,1]^2$ and possess bounded Vitali variation $V_D f$. Namely, we found exact constants C_1, C_2, and C_3 in the estimates

$$|a_{mn}(f)| \leqslant C_1 \cdot V_D f, \qquad \sum_{n=2^p+1}^{2^{p+1}} \sum_{m=2^k+1}^{2^{k+1}} |a_{mn}(f)| \leqslant C_2 \cdot V_D f$$

(Theorem 3) and in the estimate

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} |a_{mn}(f)| \leqslant C_3 \cdot V_D f$$

(Theorem 4).

Such estimates for a function of one variable were first obtained by P.L. Ul'yanov (see [1], pp. 361, 372). Exact constants for the function of one variable were found by the author in [2].

Let $D=[0,1]^2$ be the unit square, $\Pi=[a,b]\times[c,d]\subset D$ an arbitrary rectangle from D, $\tau(\Pi)$ a family of all subsets in Π of the form $T=\{(x_k,y_l)\mid k=0,\dots,m,\ l=0,\dots,n;\ a=x_0<x_1<\dots<x_m=b,\ c=y_0<y_1<\dots<y_n=d\}$. For an arbitrary point $(x,y)\in\Pi$ we call nonnegative numbers η and ξ *admissible increments in* Π with respect to x and y, respectively, if $x+\eta\in[a,b]$ and $y+\xi\in[c,d]$. If $f:\Pi\to\mathbb{R}$ is a certain function, $(x,y)\in\Pi$ and η, ξ are admissible increments in Π, then the following quantities are defined:

$$\Delta_\eta^{(1)} f(x,y) = f(x+\eta,y)-f(x,y), \quad \Delta_\xi^{(2)} f(x,y) = f(x,y+\xi)-f(x,y),$$

and

$$\Delta_\xi^{(2)}\Delta_\eta^{(1)} f(x,y) = f(x+\eta,y+\xi)+f(x,y)-f(x+\eta,y)-f(x,y+\xi).$$

Definition 1 (see [3]). By the Vitali variation of a function $f:\Pi\to\mathbb{R}$ with respect to the rectangle Π we call the quantity

$$V_\Pi f = \sup_{T\in\tau(\Pi)} \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} |\Delta_{\xi_l}^{(2)}\Delta_{\eta_k}^{(1)} f(x_k,y_l)|,$$

where $\eta_k = x_{k+1}-x_k$ and $\xi_l = y_{l+1}-y_l$ for $k=0,\dots,m-1$, $l=0,\dots,n-1$. The variation $V_\Pi f$ will be denoted by $V_{a\,c}^{b\,d} f$. The class of functions possessing a finite Vitali variation with respect to D is denoted by $V(D)$.

Lemma 1 (see [3]). *Let* $0\leqslant a\leqslant a_1\leqslant a_2\leqslant 1$ *and* $0\leqslant b\leqslant b_1\leqslant b_2\leqslant 1$. *Then*

$$V_{a\,b}^{a_2\,b_2} f = V_{a\,b}^{a_1\,b_1} f + V_{a_1\,b}^{a_2\,b_1} f + V_{a\,b_1}^{a_1\,b_2} f + V_{a_1\,b_1}^{a_2\,b_2} f.$$

Denote by $V_0(D)$ the class of functions $\{f\in V(D)\mid f(\cdot,0)\equiv 0,\ f(0,\cdot)\equiv 0\}$.