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On the Best Mean Square Approximations by Entire Functions of Exponential Type in $L_2(\mathbb{R})$ and Mean ν-Widths of Some Functional Classes

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Abstract—We consider some extremal problems of approximation theory of functions on the whole real axis $\mathbb{R}$ by entire functions of the exponential type. In particular, we find the exact values of the mean ν-widths of classes of functions, defined by the modules of continuity of the mth order ω_m and majorants Ψ satisfying the special type of restriction.

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1. In the present paper we consider the solution to a series of extremal problems of the theory of approximations of functions on the whole real axis $\mathbb{R}$ by entire functions of exponential type. A review of the obtained results in this direction can be found, for example, in [1], where one adduces a sufficiently complete comparative chronology of exact solutions to extremal problems of the indicated type, which are obtained on the classes of 2π-periodic functions in the space $L_2([0, 2\pi])$ and on the classes of functions defined on $\mathbb{R}$ and belonging to the space $L_2(\mathbb{R})$.

Recall that $L_2(\mathbb{R})$ means the space of all real measurable on $\mathbb{R}$ functions f which satisfy the condition

$$\|f\| := \|f\|_{L_2(\mathbb{R})} = \left\{ \int_{-\infty}^{\infty} |f(x)|^2 dx \right\}^{1/2} < \infty.$$

$L_2^r(\mathbb{R})$, where $r \in \mathbb{N}$, means the class of functions $f \in L_2(\mathbb{R})$, whose derivatives of the $(r-1)$th order $f^{(r-1)}$ $(f^{(0)} \equiv f)$ are locally absolutely continuous, and derivatives of the rth order $f^{(r)}$ belong to the space $L_2(\mathbb{R})$. We note that $L_2^r(\mathbb{R})$ is the Banach space with the norm $\|f\| + \|f^{(r)}\|$. We denote by $\mathbb{B}_\sigma$, where $0 < \sigma < \infty$, a narrowing on $\mathbb{R}$ of a set of all entire functions of exponential type σ, which belong to the space $L_2(\mathbb{R})$. The value

$$A_\sigma(f) := A_\sigma(f)_{L_2(\mathbb{R})} = \inf\{\|f - g_\sigma\| : g_\sigma \in \mathbb{B}_\sigma\},$$

where $f \in L_2(\mathbb{R})$, is said to be the best approximation of the function f by elements of the set $\mathbb{B}_\sigma$ in the metric of space $L_2(\mathbb{R})$. For an arbitrary class $\mathfrak{M} \subset L_2(\mathbb{R})$ we set

$$A_\sigma(\mathfrak{M}) := \sup\{A_\sigma(f) : f \in \mathfrak{M}\}.$$

A module of continuity of mth order of the function $f \in L_2(\mathbb{R})$ is the value

$$\omega_m(f, t) := \omega_m(f, t)_{L_2(\mathbb{R})} = \sup\{\|\Delta_h^m(f)\| : |h| \leqslant t\},$$

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