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The Berger—Ebin Theorem and Harmonic Maps and Flows

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Abstract—The purpose of this paper is to present a geometrization of the Berger—Ebin theorem. We use this theorem for the study of harmonic maps and flows, in particular, Ricci solitons. We also clarify the role of a vector field in the corresponding decompositions.

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1. INTRODUCTION

1.1. The Berger—Ebin theorem [1] is a classical result in the Riemannian geometry in the large and is included in monographs (see, e.g., [2]). By this theorem, an arbitrary symmetric 2-form φ on a closed oriented Riemannian manifold can be represented as the orthogonal sum $\varphi = \varphi^0 + \varphi^1$ of a divergence-free symmetric 2-form φ^0 and a symmetric 2-form $\varphi^1 := L_X g$, where L_X is the Lie derivative with respect to a vector field X.

In this paper, we use the Berger—Ebin decomposition for the study of harmonic maps and flows, in particular, Ricci solitons. We also clarify the role of the vector field X in the corresponding decompositions.

2. THE BERGER—EBIN DECOMPOSITION

2.1. Let (M, g) be a C^∞ Riemannian manifold of dimension $n \geq 2$, and let ∇ denote the Levi-Civita connection of (M, g). Let TM denote the tangent bundle of M, T^*M the cotangent bundle, and S^pM the bundle of symmetric p-forms on M. The metrics on the fibers of these bundles are determined by the Riemannian structure g on M. Denote by $C^\infty TM$, $C^\infty T^*M$, and $C^\infty S^pM$ the spaces of C^∞-sections of the above listed bundles respectively. These spaces possess natural scalar products $\langle \cdot, \cdot \rangle$. In particular ([3], Chap. 4, § 1), we have $\langle \varphi, \varphi' \rangle := \int_M \frac{1}{p!} g(\varphi, \varphi') dv$ for arbitrary $\varphi, \varphi' \in C^\infty S^pM$.

Define a differential operator $\delta^* : C^\infty S^pM \to C^\infty S^{p+1}M$ by the relation

$$(\delta^*\varphi)_{i_0 i_1 \dots i_{p-1} i_p} = \nabla_{i_0} \varphi_{i_1 \dots i_{p-1} i_p} + \dots + \nabla_{i_p} \varphi_{i_0 i_1 \dots i_{p-1}}$$

for the local components $\varphi_{i_1 \dots i_p}$ of an arbitrary form $\varphi \in C^\infty S^pM$ and the covariant differentiation ∇_j with respect the vector fields $X_j = \frac{\partial}{\partial x^j}$ in terms of local coordinates $x^1, x^2, \dots, x^n$ on M. In particular, for an arbitrary 1-form ω and a vector field $X := \omega^\#$ dual to the 1-form ω with respect to g, we have $\delta^*\omega = L_X g$, where L_X denotes the Lie derivative with respect to a vector field X ([1]; [2], pp. 54–55). In the classical tensor notation, the symbol # corresponds to the raising of an index ([2], P. 47). In particular, $\delta^*\omega = 0$ if and only if $L_X g = 0$. In this case, the local one-parameter group generated by X consists of local isometries, and X is a *Killing vector field* ([2], P. 61).

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