ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2011, Vol. 55, No. 5, pp. 70–73. © Allerton Press, Inc., 2011.
Original Russian Text © M.B. Karmanova, 2011, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2011, No. 5, pp. 84–87.

An Example of a Carnot Manifold with C^1-Smooth Basis Vector Fields

M. B. Karmanova[1*]
(Submitted by S.K. Vodop'yanov)

[1] *Sobolev Institute of Mathematics, Siberian Branch of Russian Academy of Sciences, pr. Ak. Koptyuga 4, Novosibirsk, 630090 Russia*
Received November 22, 2010

Abstract—We construct an example of a Carnot manifold whose basis vector fields are of class C^1 but not of class C^2.

DOI: 10.3103/S1066369X11050124

Keywords and phrases: *Carnot manifold, commutator of vector fields.*

In the paper, we give an example of a Carnot manifold $(\mathbb{M}, X_1, X_2, X_3)$ whose basis vector fields are only of class C^1. As a manifold $\mathbb{M}$, we take a neighborhood in $\mathbb{R}^3$.

It is known that a Carnot–Carathéodory space $\mathbb{M}$ is a connected Riemannian manifold with a given subbundle of the tangent bundle $T\mathbb{M}$, called the horizontal subbundle, which generates with the use of commutation operation the entire tangent bundle $T\mathbb{M}$. Such a description is correct only for sufficiently smooth vector fields belonging to the horizontal subbundle. In the case when horizontal fields are only of class C^1, another definition takes place [1], which is equivalent to the classical one in the smooth case. The study of Carnot manifolds with basis vector fields of minimal possible smoothness is motivated by applications of the theory of Carnot manifolds in neurobiology and robotechnics. For example, it is known [2–4] that the principle in accordance with which the human brain constructs a part of a closed black and white image is based on the properties of minimal surfaces in a three-dimensional Carnot manifold. Manifolds whose basis vector fields are of minimal smoothness were studied in [1, 5–10] (for the applications of the results of the cited papers, see [11–15]. In particular, the following facts, which are new in the case of basis vector fields of minimal smoothness, have been established (for smooth vector fields, these results are classical): A generalized triangle inequality for d_∞, the Rashevskii–Chow theorem [16, 17], the Ball–Box theorem on comparison of the Carnot–Carathéodory metric and the quasimetric d_∞ [1], the Mitchell theorem [18] on the convergence of Carnot–Carathéodory spaces rescaled with respect to a point $u \in \mathbb{M}$ to a nilpotent tangent cone, the local Gromov approximation theorem, and other.

Definition (cf. [1, 19, 20]). Fix a connected N-dimensional Riemannian C^∞-manifold $\mathbb{M}$. This manifold $\mathbb{M}$ is called a *Carnot–Carathéodory space* if, in the tangent bundle $T\mathbb{M}$, there are given a subbundle $H\mathbb{M}$ and a filtration of $T\mathbb{M}$ such that $H\mathbb{M} = H_1 \subseteq H_2 \subseteq \cdots \subseteq H_M = T\mathbb{M}$, where $\dim H_i = \text{const}$ for all $i = 1, \dots, M$, and, for each point $p \in \mathbb{M}$, there exists a neighborhood $U \subset \mathbb{M}$ with C^1-smooth vector fields $X_1, \dots, X_N$ on U such that, at each point $v \in U$:

1) $X_1(v), \dots, X_N(v)$ is a basis of $T_v\mathbb{M}$;

2) $H_i(v) = \operatorname{span}\{X_1(v), \dots, X_{\dim H_i}(v)\}$ is a subspace of $T_v\mathbb{M}$ of dimension $\dim H_i$;

3) $[X_i, X_j](v) = \sum\limits_{k:\deg X_k \le \deg X_i + \deg X_j} c_{ijk}(v) X_k(v)$, where the *degree* $\deg X_k$ is defined to be $\min\{m \mid X_k \in H_m\}$.

A Carnot–Carathéodory space with the property that

[*] E-mail: maryka@math.nsc.ru.