ISSN 1066-369X, Russian Mathematics, 2017, Vol. 61, No. 2, pp. 49–54. © Allerton Press, Inc., 2017.
Original Russian Text © V.M. Savchin, S.A. Budochkina, 2017, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2017, No. 2, pp. 58–64.

Invariance of Functionals and Related Euler–Lagrange Equations

V. M. Savchin* **and S. A. Budochkina****

Peoples' Friendship University of Russia
ul. Miklukho-Maklaya 6, Moscow, 117198 Russia
Received September 4, 2015

Abstract—We establish a connection between symmetries of functionals and symmetries of the corresponding Euler–Lagrange equations. A similar problem is investigated for equations with quasi-B_u-potential operators.

DOI: 10.3103/S1066369X17020062

Keywords: *Hamiltonian action, invariance, symmetry, Lie algebra.*

Introduction

In the book by Olver ([1], P. 330) a theorem is given: the symmetry group of a functional of the form

$$F[u] = \int_{\Omega} f(x, u_\alpha)dx \tag{0.1}$$

is also the symmetry group of the related Euler–Lagrange equation.

It is known [2] that there exist boundary-value problems not admitting variational formulations in the class of functionals of the form (0.1), but at the same time we can reduce them to variational form using so-called non-Euler classes of functionals.

Note that symmetry is widely used in qualitative analysis of finite-dimensional systems [3, 4]. Development of such ideas for infinite-dimensional systems is an actual and interesting problem.

The aim of the paper is to prove that the symmetry group of an arbitrary Gâteaux differentiable functional is the symmetry group for the corresponding Euler–Lagrange equations and also to establish that in the general case non-linear generators of symmetry groups produce a Lie algebra with respect to the operation of commutator of two generators.

We will follow the notation and terminology given in [5–7].

1. FUNCTIONAL SYMMETRIES AND RELATED ALGEBRAIC STRUCTURES

Consider the operator equation

$$N(u) = 0, \quad u \in D(N), \tag{1.1}$$

where $N : D(N) \subset U \to V$ is a Gâteaux differentiable operator, U and V are linear normed vector spaces over the field of real numbers $\mathbb{R}$, $D(N)$ is the domain of definition of the operator N. We will require that $D(N)$ is a convex set, $\overline{D(N)} = U$, and $U \subseteq V$.

Assume that the operator N of problem (1.1) is potential on $D(N)$ with respect to a fixed continuous bilinear form

$$\langle \cdot, \cdot \rangle : V \times U \to \mathbb{R}. \tag{1.2}$$

*E-mail: vsavchin@yandex.ru.
**E-mail: sbudotchkina@yandex.ru.