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Behavior of a Singular Integral with Hilbert Kernel at a Point of Weak Continuity of its Density

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Abstract—We study properties of a singular integral with the Hilbert kernel at a fixed point, where the modulus of continuity of its density has logarithmic order, and the integral is not necessarily convergent.

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We consider a singular (in the sense of the principal value) integral with the Hilbert kernel

$$I(\gamma_0) = \int_0^{2\pi} \phi(\gamma) \cot \frac{\gamma - \gamma_0}{2} d\gamma, \tag{1}$$

where the density $\phi(\gamma)$ is a continuous function defined in the interval $[0, 2\pi]$, $\gamma_0 \in [0, 2\pi]$, and $\phi(0) = \phi(2\pi)$. As is known ([1], pp. 18, 46), this integral converges at the point γ_0 if the function $\phi(\gamma)$ satisfies the Hölder condition (condition H) at this point. If the density $\phi(\gamma)$ satisfies condition H in some part of the interval $[0, 2\pi]$, then by virtue of the Plemelj–Privalov theorem ([1], pp. 59, 61) the integral $I(\gamma_0)$ satisfies condition H in any segment lying inside the mentioned part. In the case, when the density $\phi(\gamma)$ has an integrable singularity, the behavior of integral (1) is described in [1] (pp. 95, 160) and [2]. The connection of continuity modules of a singular integral and its density is considered in [3].

If at a fixed point $\gamma = c$ of the interval $(0, 2\pi)$ a continuous function $\phi(\gamma)$ does not satisfy condition H, then integral (1) can be divergent there, and in this case one has to study its behavior for $\gamma_0 \to c$.

In the present paper we consider the latter question under the assumption that in a segment $[c^-, c^+]$ containing a point $\gamma = c$ (we assume that the length of this segment is sufficiently small, namely, for a time being we assume that it is less than one; later we will determine it more precisely) the density $\phi(\gamma)$ is representable as

$$\phi(\gamma) = \frac{\Phi(\gamma)}{\left|\ln \sin^2 \frac{\gamma - c}{2}\right|},$$

where $\Phi(\gamma)$ is a given function satisfying condition H in each of intervals $[c^-, c]$ and $[c, c^+]$ with unequal, generally speaking, one-sided limits $\Phi(c-0)$, $\Phi(c+0)$. The behavior of the integral $I(\gamma_0)$ at points c^-, c^+ needs no special consideration if the density $\phi(\gamma)$ satisfies condition H near these points.

For simplicity, we put $c^- > 0$ and $c^+ < 2\pi$, and represent integral (1) in the form

$$I(\gamma_0) = \left(\int_0^{c^-} + \int_{c^+}^{2\pi}\right) \phi(\gamma) \cot \frac{\gamma - \gamma_0}{2} d\gamma + \int_{c^-}^{c^+} \frac{\Phi(\gamma) - \Phi(c \pm 0)}{\left|\ln \sin^2 \frac{\gamma - c}{2}\right|} \cot \frac{\gamma - \gamma_0}{2} d\gamma$$
$$+ \int_{c^-}^{c^+} \frac{\Phi(c \pm 0)}{\left|\ln \sin^2 \frac{\gamma - c}{2}\right|} \cot \frac{\gamma - \gamma_0}{2} d\gamma = J_1(\gamma_0) + J_2(\gamma_0) + J_3(\gamma_0) + J_4(\gamma_0), \quad \gamma_0 \neq c, \tag{2}$$

here and in what follows the upper signs must be taken for $\gamma > c$, and lower ones for $\gamma < c$.

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