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LÉVY LAPLACIANS AND ANNIHILATION PROCESS

B.O. Volkov

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, 119991 Russia

Bauman Moscow State Technical University, Moscow, 105005 Russia

Abstract

The Lévy Laplacians are infinite-dimensional Laplace operators defined as the Cesaro mean of the second-order directional derivatives. In the theory of Sobolev–Schwarz distributions over a Gaussian measure on an infinite-dimensional space (the Hida calculus), we can consider two canonical Lévy Laplacians. The first Laplacian, the so-called classical Lévy Laplacian, has been well studied. The interest in the second Laplacian is due to its connection with the Malliavin calculus (the theory of Sobolev spaces over the Wiener measure) and the Yang–Mills gauge theory. The representation in the form of the quadratic function of the annihilation process for the classical Lévy-Laplacian is known. This representation can be obtained using the S-transform (the Segal–Bargmann transform). In the paper, we show, by analogy, that the representation in the form of the quadratic function of the derivative of the annihilation process exists for the second Lévy-Laplacian. The obtained representation can be used for studying the gauge fields and the Lévy Laplacian in the Malliavin calculus.

Keywords: Lévy Laplacian, Hida calculus, quantum probability, annihilation process

Introduction

In the present paper, we study some relationships between infinite-dimensional Laplacians and quantum stochastic processes.

Let us recall the definition of the Lévy Laplacian. Let E be a real locally convex space continuously embedded into a separable Hilbert space H. Let the image of E under the embedding be dense in H. The value of the Lévy Laplacian on a function f on E is determined by the formula

$$\Delta_L f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \langle f''(x)e_k, \quad e_k\rangle, \tag{1}$$

where $\{e_n\}$ is an orthonormal basis in H such that all its elements belong to E. This definition depends on a choice of the orthonormal basis $\{e_n\}$. For so-called weakly uniformly dense bases in $H = L_2([0,1],\mathbb{R})$, this definition coincides with the definition of the Lévy Laplacian as an integral functional determined by the special form of the second derivative (see [1, 2]). If we replace in (1) the Césaro mean by the sum of the series, we obtain the definition of the Volterra-Gross Laplacian.

Now let

$$E = W_0^{1,2}([0,1],\mathbb{R}) := \{\gamma \in AC([0,1],\mathbb{R})\colon \gamma(0) = 0, \dot\gamma \in L_2([0,1],\mathbb{R})\}.$$

This space is canonically isomorphic to $L_2([0,1],\mathbb{R})$ (the isomorphism is determined by the differentiation). We can define the Lévy Laplacian on the functions on E by the following two ways. On the one hand, we can choose some good orthonormal basis in $W_0^{1,2}([0,1],\mathbb{R})$. Then we obtain the Lévy Laplacian $\Delta_L^{(1)}$ of order 1. On the other hand, $W_0^{1,2}([0,1],\mathbb{R})$ is embedded into $H = L_2([0,1],\mathbb{R})$. We can choose some good orthonormal basis in H and obtain the Lévy Laplacian $\Delta_L^{(-1)}$ of order (-1).[1]

The theory of the Sobolev–Schwartz distributions over the abstract Wiener measure is called the Hida calculus or the white noise theory. It is known that the Volterra–Gross Laplacian Δ_{VG} can be represented by the formula $\Delta_{VG} = \int\limits_{\mathbb{R}} a_t^2\,dt$, where a is the annihilation process (see, e.g., [7, 8]). An analogue of the Lévy Laplacian Δ_L^1 in the Hida calculus we will call the classical Lévy Laplacian. The classical Lévy Laplacian Δ_L^1 can be interpreted as $\int\limits_0^1 a_t^2(dt^2)$ (see [8] and also [9], where this formula appears with the reference to Kuo). In this paper, we will show that the analogue of the Lévy Laplacian $\Delta_L^{(-1)}$ can be interpreted as $\dfrac{1}{\pi^2}\int\limits_0^1 \dot a_t^2(dt^2)$. The connections between the Lévy Laplacians and quantum stochastic processes were also discussed in [5, 6, 10, 11]. In the latter paper, a rigorous meaning for the formulas

$$\Delta_L = \lim_{\varepsilon\to 0}\int\limits_{|s-t|<\varepsilon} a_s a_t\,ds\,dt$$

and

$$\Delta_L^{(-1)} = \frac{1}{\pi^2}\lim_{\varepsilon\to 0}\int\limits_{|s-t|<\varepsilon} \dot a_s \dot a_t\,ds\,dt$$

was given.

Note that one of the main reasons for the interest in the Lévy Laplacian $\Delta_L^{(-1)}$ is its connection to the gauge fields (see [4, 12–16]).

1. Lévy Laplacians

$\{p_1,\ldots,p_d\}$ is an orthonormal basis in $\mathbb{R}^d$ everywhere below. If E is a locally convex space (LCS), its dual space E^* is equipped with strong topology. If E and V are LCSs, the space $L^b(E,V)$ is the space of all continuous linear operators from E to V. We assume that $L^b(E,V)$ is equipped with the topology of the uniform convergence on bounded sets.

Let $E_{\mathbb{C}} = S(\mathbb{R},\mathbb{C}^d)$ be the Schwartz space of $\mathbb{C}^d$-valued rapidly decreasing functions and $E_{\mathbb{C}}^* = S^*(\mathbb{R},\mathbb{C}^d)$ be the space of generalized functions of slow growth. Let $E_\mu = \{\xi = (\xi^1,\ldots,\xi^d)\in E_{\mathbb{C}}\colon \xi^\nu = 0, \text{if } \mu\neq\nu\}$. Then $E_{\mathbb{C}} = E_1\oplus\ldots\oplus E_d$. Let $T_2 = T_2(\mathbb{R}^d,\mathbb{C})$ and T_2^{sym} be the space of all $\mathbb{C}$-valued tensors of type $(0,2)$ and the space of all symmetric $\mathbb{C}$-valued tensors of type $(0,2)$ on $\mathbb{R}^d$, respectively. Let $L_2^{\mathrm{sym}}(\mathbb{R}^2,T_2) = \{g\in L_2(\mathbb{R}^2,T_2)\colon g_{\mu\nu}(t,s) = g_{\nu\mu}(s,t)\}$. Let $C_L^2(E_{\mathbb{C}},\mathbb{C})$ be the space of all two times Fréchet complex differentiable $\mathbb{C}$-valued functions on $E_{\mathbb{C}} = S(\mathbb{R},\mathbb{C}^d)$ satisfying the following condition:

the second derivative of $f\in C_L^2(E_{\mathbb{C}},\mathbb{C})$ has the form

$$\langle f''(\xi)\zeta,\eta\rangle = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} K_{\mu\nu}^V(\xi;s,t)\zeta^\mu(t)\eta^\nu(s)\,dt\,ds + \int\limits_{\mathbb{R}} K_{\mu\nu}^L(\xi;t)\zeta^\mu(t)\eta^\nu(t)\,dt,\ \zeta,\,\eta\in E_{\mathbb{C}},\quad (2)$$

[1] One can consider the family of so-called exotic Lévy Laplacian $\Delta_L^{(l)}$, where $l\geq 0$ (see [3]). The Laplacian $\Delta_L^{(-1)}$ belongs to the extension of this family for negative $l<0$ (see [4] and also [5, 6]).

where $K^V(\xi,\cdot,\cdot)\in L_2^{sym}(\mathbb{R}^2,T_2)$, $K^L(\xi,\cdot)\in L_\infty(\mathbb{R},T_2^{sym})$ for any $\xi\in E_{\mathbb{C}}$. (K^V is the Volterra kernel and K^L is the Lévy kernel of the second derivative).

If $f\in C_L^2(E_{\mathbb{C}},\mathbb{C})$, it is possible to extend $f''(\xi)$ as a bilinear jointly continuous functional on $L_2(\mathbb{R},\mathbb{C}^d)\times L_2(\mathbb{R},\mathbb{C}^d)$. We will denote this extension by the same symbol.

Let $\{e_n\}$ be an orthonormal basis in $L_2([0,1],\mathbb{R})$. We identify any element $h\in L_2([0,1],\mathbb{R})$ with $\mathbf{h}\in L_2(\mathbb{R},\mathbb{R})$ defined by

$$\mathbf{h}(t)=\begin{cases} h(t), & \text{if } t\in[0,1],\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

We do not require $\{e_n\}$ to have its elements from $S(\mathbb{R},\mathbb{R})$.

Definition 1. The Lévy Laplacian $\Delta_L^{\{e_n\},s}$ of order $s\in\{-1,1\}$ is a linear mapping from $\operatorname{Dom}\Delta_L^{\{e_n\},s}$ to the space of all $\mathbb{C}$-valued functions on $E_{\mathbb{C}}$ defined by:

$$\Delta_L^{\{e_n\},s}f(\xi)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{\mu=1}^{d}k^{1-s}\langle f''(\xi)p_\mu e_k,p_\mu e_k\rangle, \tag{4}$$

where $\operatorname{Dom}\Delta_L^{\{e_n\},s}$ is the space of all f unctions $f\in C_L^2(E_{\mathbb{C}},\mathbb{C})$, for which the right side of (4) exists for any $\xi\in E_{\mathbb{C}}$.

The following definition is from [1].

Definition 2. An orthonormal basis $\{e_n\}$ in $L_2([0,1],\mathbb{R})$ is weakly uniformly dense, if

$$\lim_{n\to\infty}\int_0^1 h(t)\Big(\frac{1}{n}\sum_{k=1}^{n}e_k^2(t)-1\Big)dt=0$$

for any $h\in L_\infty([0,1],\mathbb{R})$.

Let $l_n(t)=\sqrt{2}\sin(\pi nt)$ and $h_n(t)=\sqrt{2}\cos(\pi nt)$ for $n\in\mathbb{N}$ and $h_0(t)=1$. The orthonormal bases $\{h_n\}_{n=1}^\infty$ and $\{l_n\}_{n=0}^\infty$ in $L_2([0,1],\mathbb{R})$ are weakly uniformly dense.

Proposition 1. *Let $\{e_n\}$ be a weakly uniformly dense basis in $L_2([0,1],\mathbb{R})$. Let $f\in C_L^2(E_{\mathbb{C}},\mathbb{C})$. Then*

$$\Delta_L^{\{e_n\},1}f(\xi)=\sum_{\mu=1}^{d}\int_0^1 K_{\mu\mu}^L(\xi,t)\,dt. \tag{5}$$

This is a well-known fact for $d=1$ (see, e.g., [1, 2, 7]). If $d>1$ formula (5) can be proved by analogy (see [4]).

Due to the kernel theorem (see, e.g., [17]) for any $f\in C^2(E_{\mathbb{C}},\mathbb{C})$ the second partial derivative $f''_{E_\mu E_\mu}(\xi)$ belongs to $S^*(\mathbb{R}^2,\mathbb{C})$. Let $C_{L,(-1)}^2(E_{\mathbb{C}},\mathbb{C})$ be the space of all $f\in C_L^2(E_{\mathbb{C}},\mathbb{C})$ such that for any $\mu\in\{1,\dots,d\}$ and for any $\xi\in E_{\mathbb{C}}$ the second mixed generalized derivative of $f''_{E_\mu E_\mu}(\xi)\in S^*(\mathbb{R}^2,\mathbb{C})$ has the form

$$\Big\langle\frac{\partial^2}{\partial t\partial s}(f''_{E_\mu E_\mu}(\xi)),\phi\Big\rangle=\int_{\mathbb{R}^2}\phi(s,t)\nu_{\mu\mu}^{\xi}(ds\,dt),\quad \phi\in S(\mathbb{R}^2,\mathbb{C}), \tag{6}$$

where $\nu_{\mu\mu}^{\xi}$ is a σ-additive σ-finite $\mathbb{C}$-valued measure on $\mathbb{R}^2$.

Let $I=\{(s,t)\in\mathbb{R}^2\colon s=t,0<s<1\}$. Let $\mathbf{1}_I$ be the indicator of this set.

Proposition 2. *If* $f \in C^2_{L,(-1)}(E_{\mathbb{C}},\mathbb{C})$, *then*

$$\Delta_L^{\{h_n\},-1} f(\xi) = \sum_{\mu=1}^{d} \frac{1}{\pi^2} \int_0^1 \boldsymbol{1}_I(s,t)\nu^{\xi}_{\mu\mu}(dsdt). \tag{7}$$

Proof. Let

$$\tau(t) = \begin{cases} \exp\Big(\dfrac{1}{t^2-1}\Big), & \text{if } |t|<1,\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

Let $\tau_1(t) = \tau(t)\Big/\int\limits_{-\infty}^{\infty}\tau(t)\,dt$ and $\tau^{\varepsilon}(t) = \tau_1(t/\varepsilon)/\varepsilon$. Let $l^{\varepsilon}_n = l_n * \tau^{\varepsilon}$. Then $l^{\varepsilon}_n \in S(\mathbb{R},\mathbb{R})$, the support of l^{ε}_n belongs to $[-\varepsilon, 1+\varepsilon]$ and l^{ε}_n converges to l_n as $\varepsilon \to 0$ uniformly on any compact set. Let $h^{\varepsilon}_n = h_n * \tau^{\varepsilon}$. Then h^{ε}_n converges to h_n in $L_2(\mathbb{R},\mathbb{R})$ as $\varepsilon \to 0$. Due to $l'_n = \pi n h_n$, we have $(l^{\varepsilon}_n)' = \pi n h^{\varepsilon}_n$. Hence, we obtain

$$\begin{aligned}\pi^2 n^2 \langle f''_{E_\mu E_\mu}(\xi), h_n \otimes h_n\rangle &= \lim_{\varepsilon\to 0} \pi^2 n^2 \langle f''_{E_\mu E_\mu}(\xi), h^{\varepsilon}_n \otimes h^{\varepsilon}_n\rangle = \\ &= \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^2} l^{\varepsilon}_n(s) l^{\varepsilon}_n(t)\nu^{\xi}_{\mu\mu}(dt\,ds) = \int\limits_{\mathbb{R}^2} l_n(s)l_n(t)\nu^{\xi}_{\mu\mu}(dt\,ds).\end{aligned} \tag{9}$$

The last equality is due to Lebesgue's dominated convergence theorem. For any $(s,t) \in \mathbb{R}^2$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} l_n(s)l_n(t) = \mathbf{1}_I(s,t) \tag{10}$$

and

$$\sup_{n\in\mathbb{N}} \Big|\frac{1}{n}\sum_{k=1}^{n} l_n(s)l_n(t)\Big| \le 2. \tag{11}$$

Thus Lebesgue's dominated convergence theorem and (9) together imply (7). □

Remark 1. One of the approaches to define the Lévy Laplacian is to define it as the integral functional determined by the special form of the second derivative (see [1, 2, 7]). Particularly, the operators $\Delta_L^{(1)}$ and $\Delta_L^{(-1)}$ can be defined by formulas (5), and (7), respectively. It can be shown that form (6) is generalization of the form of the second derivative from [12].

2. Lévy Laplacians in Hida calculus

The operator

$$\mathbf{A} = 1 + t^2 - \frac{d^2}{dt^2},$$

is a self-adjoint operator on $H_{\mathbb{C}} = L_2(\mathbb{R},\mathbb{C}^d)$. For any $p \ge 0$ let E_p be the domain of $\mathbf{A}^p$ equipped with the Hilbert norm $|\xi|_p = |\mathbf{A}^p\xi|_{H_{\mathbb{C}}}$. For any $p < 0$ let E_p be the completion of $H_{\mathbb{C}}$ with respect with the Hilbert norm $|\xi|_p = |\mathbf{A}^p\xi|_{H_{\mathbb{C}}}$. Then $S(\mathbb{R},\mathbb{C}^d)$ coincides with the projective limit $\operatorname{proj\,lim}_{p\to+\infty} E_p$ and $S^*(\mathbb{R},\mathbb{C}^d)$ coincides with the inductive limit $\operatorname{ind\,lim}_{p\to+\infty} E_{-p}$. We have the real and complex Gelfand triplets:

$$S(\mathbb{R},\mathbb{R}^d) = E_{\mathbb{R}} \subset L_2(\mathbb{R},\mathbb{R}^d) = H_{\mathbb{R}} \subset S^*(\mathbb{R},\mathbb{R}^d) = E^*_{\mathbb{R}}.$$

and

$$E_{\mathbb{C}} \subset H_{\mathbb{C}} \subset E_{\mathbb{C}}^*.$$

The Fock space over the Hilbert space E_p is defined as

$$\Gamma(E_p) = \Big\{\phi = (f_n)_{n=0}^{\infty};\ f_n \in E_p^{\widehat{\otimes} n},\ \|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|f_n|_p^2 < \infty\Big\}.$$

Let $\mathcal{E} = \operatorname{proj}\lim_{p\to+\infty}\Gamma(E_p)$. Then $\mathcal{E}^* = \operatorname{ind}\lim_{p\to+\infty}\Gamma(E_{-p})$. Let the symbol $\langle\langle\cdot,\cdot\rangle\rangle$ denote the duality form on $\mathcal{E}^*\times\mathcal{E}$.

Let μ_I be the Gaussian pseudomeasure on $S(\mathbb{R},\mathbb{R}^d)$ with the Fourier transform $\widetilde{\mu}_I(\xi) = \exp(-(\xi,\xi)_{H_{\mathbb{R}}}/2)$. The Minlos–Sazonov theorem implies that μ_I is σ-additive measure on $S(\mathbb{R},\mathbb{R}^d)$. The unitary Wiener-Itô–Segal isomorphism between $\Gamma(H_{\mathbb{C}})$ and $L_2(E_{\mathbb{R}}^*,\mu_I,\mathbb{C})$ is determined by the values of this isomorphism on the coherent states:

$$\psi_\xi = \Big(1,\xi,\frac{\xi^{\otimes 2}}{2},\dots,\frac{\xi^{\otimes n}}{n!},\dots\Big) \longleftrightarrow \psi_\xi = e^{\langle x,\xi\rangle - \langle\xi,\xi\rangle/2},\quad \xi\in E_{\mathbb{C}}.$$

Below we will not distinguish between the spaces $\Gamma(H_{\mathbb{C}})$ and $L_2(E_{\mathbb{R}}^*,\mu_I,\mathbb{C})$. The Gelfand triplet $\mathcal{E}\subset L_2(E_{\mathbb{R}}^*,\mu_I,\mathbb{C})\subset\mathcal{E}^*$ is called the Hida–Kubo–Takenaka space. $\mathcal{E}$ is the space of white noise test functionals (Hida test functionals), and $\mathcal{E}^*$ is the space of white noise generalized functionals (Hida generalized functionals).

The S-transform (the Bargman–Segal transform) of generalized white noise functional $\Phi\in\mathcal{E}^*$ is the function $S\Phi\colon E_{\mathbb{C}}\to\mathbb{C}$ defined by the formula $S\Phi(\xi) = \langle\langle\Phi,\psi_\xi\rangle\rangle$, $\xi\in E_{\mathbb{C}}$. It is known that a complex function G on $E_{\mathbb{C}}$ is a S-transform of some generalized white noise functional if and only if G satisfies the following two conditions (see, e.g., [7, 18]):

A. for any $\zeta,\eta\in E_{\mathbb{C}}$ the function $G_{\zeta,\eta}(z) = G(z\eta+\zeta)$ is entire on $\mathbb{C}$;

B. there exist constants $C_1, C_2 > 0$ and $p > 0$ that for each $\xi\in E_{\mathbb{C}}$ hold:

$$|G(\xi)| \le C_1\exp(C_2|\xi|_p^2).$$

If a complex-valued function on the space $E_{\mathbb{C}}$ satisfies the conditions above, it is called U-functional. Let the symbol $\mathcal{F}_U$ denote the space of all U-functionals.

Definition 3. The domain of the Lévy Laplacian $\widetilde{\Delta}_L^{\{e_n\},s}$ of order $s\in\{-1,1\}$ is the space $\operatorname{Dom}\widetilde{\Delta}_L^{\{e_n\},s} = \{\Phi\in\mathcal{E}^*\colon S\Phi\in\operatorname{Dom}\Delta_L^{\{e_n\},s}, \Delta_L^{\{e_n\},s}S\Phi\in\mathcal{F}_U\}$. The Lévy Laplacian $\widetilde{\Delta}_L^{\{e_n\},s}$ is a linear mapping from $\operatorname{Dom}\widetilde{\Delta}_L^{\{e_n\},s}$ to $\mathcal{E}^*$ defined by

$$\widetilde{\Delta}_L^{\{e_n\},s}\Phi = S^{-1}\Delta_L^{\{e_n\},s}(S\Phi). \tag{12}$$

Remark 2. It is known that if $\Phi\in L_2(E_{\mathbb{R}}^*,\mu_I,\mathbb{C})$, then $S\Phi\in C_L^2(E_{\mathbb{C}},\mathbb{C})$ and the Lévy kernel of $S\Phi''$ is equal to zero (see, e.g., theorem 6.42 from [18]). Hence, if $\Phi\in L_2(E_{\mathbb{R}}^*,\mu_I,\mathbb{C})$, then $\widetilde{\Delta}_L^{\{e_n\},1}\Phi = 0$, where $\{e_n\}$ is a weakly uniformly dense basis.

The elements from $\mathcal{E}$ can be realized as entire functions on $E_{\mathbb{R}}^*$ (see, e.g., [7]). For any $\zeta\in E_{\mathbb{R}}^*$ let $a(\zeta)$ be the operator of differentiation in the direction:

$$a(\zeta)\phi(\xi) = \lim_{t\to 0}(\phi(\xi+t\zeta)-\phi(\xi))/t,\quad \xi\in E_{\mathbb{R}}^*, \phi\in\mathcal{E}.$$

If $\zeta \in E^*_{\mathbb{R}}$, then $a(\zeta) \in L^b(\mathcal{E},\mathcal{E})$. If $\zeta \in E_{\mathbb{R}}$, then $a(\zeta)$ can be extended to the operator $\widetilde{a}(\zeta) \in L^b(\mathcal{E}^*,\mathcal{E}^*)$. For any $\Phi \in \mathcal{E}^*$ and $\zeta \in E_{\mathbb{R}}$ the following holds:

$$S(\widetilde{a}(\zeta)\Phi)(\xi) = \langle S\Phi'(\xi), \zeta\rangle.$$

A continuous mapping from $\mathbb{R}$ to $L^b(\mathcal{E},\mathcal{E}^*)$ is a quantum stochastic process in the sense of the Hida calculus (see [19]). Note that there is the canonical embedding id of $\mathcal{E}^*$ into $L(\mathcal{E},\mathcal{E}^*)$ defined by the following way:

$$(id\Phi)(\phi) = \Phi\phi, \Phi \in \mathcal{E}^*, \quad \phi \in \mathcal{E}.$$

The mapping $\mathbb{R} \ni t \to a^\mu_t = a(p_\mu\delta_t) \in L^b(\mathcal{E},\mathcal{E}^*)$ is a quantum stochastic process, which is called the annihilation process. The mapping $\mathbb{R} \ni t \mapsto (a^\mu_t)^* = a(p_\mu\delta_t)^* \in L^b(\mathcal{E},\mathcal{E}^*)$ is a quantum stochastic process, which is called the creation process. It is possible to show that these mappings are smooth and the mapping $\mathbb{R} \ni t \mapsto \dot{a}^\mu_t \in L^b(\mathcal{E},\mathcal{E}^*)$ is also quantum stochastic process (see [20]). The sum $(a^\mu_t + (a^\mu_t)^*)$ is the white noise process, which is the derivative of the Brownian motion.

Using Obata's result on the integral kernel operators (see [17, 20]), it is possible to give a rigorous sense for the integral $\int_0^1\int_0^1 \kappa_{\mu\nu}(t,s)a^\mu_t a^\nu_s dtds$, where $\kappa \in E^{\otimes 2}_{\mathbb{C}}$. If $\phi,\, \varphi \in \mathcal{E}$, then

$$\eta_{\phi,\varphi}(s,t) = (\eta^{\mu\nu}_{\phi,\varphi}(s,t)) = (\langle\langle a^\mu_s a^\nu_t \phi, \varphi\rangle\rangle) \in E^{\otimes 2}_{\mathbb{C}}.$$

If $\kappa \in (E^{\otimes 2}_{\mathbb{C}})^*$, then there exists the unique $\Xi_{0,2}(\kappa) \in L^b(\mathcal{E},\mathcal{E})$ such that $\langle\langle \Xi_{0,2}(\kappa)\phi, \varphi\rangle\rangle = \langle \kappa, \eta_{\phi,\varphi}\rangle$. If $\kappa \in E^{\otimes 2}_{\mathbb{C}}$, then $\Xi_{0,2}(\kappa)$ can be extended to the operator $\widetilde{\Xi}_{0,2}(\kappa) \in L^b(\mathcal{E}^*,\mathcal{E}^*)$. For any $\Phi \in \mathcal{E}^*$ and $\kappa \in E^{\otimes 2}_{\mathbb{C}}$ the following holds:

$$S\Big(\int_0^1\int_0^1 \kappa_{\mu\nu}(t,s)a^\mu_t a^\nu_s\, dt\, ds\, \Phi\Big)(\xi) = S(\widetilde{\Xi}_{0,2}(\kappa)\Phi)(\xi) = \langle S\Phi''(\xi), \kappa\rangle. \tag{13}$$

Let $\{e_n\}$ be a weakly uniformly dense basis. If $\Phi \in \mathrm{Dom}\widetilde{\Delta}^{\{e_n\},1}_L$ and $S\Phi \in C^2_L(E_{\mathbb{C}},\mathbb{C})$, proposition 1 implies

$$S(\widetilde{\Delta}^{\{e_n\},1}_L\Phi)(\xi) = \sum_{\mu=1}^d \int_0^1 K^L_{\mu\mu}(\xi,t)\, dt = \sum_{\mu=1}^d \langle S\Phi''_{E_\mu E_\mu}(\xi), \mathbf{1}_I\rangle. \tag{14}$$

So, the right side of (13) has sense if $\kappa_{\mu\nu} = \delta_{\mu\nu}\mathbf{1}_I$, where $\delta_{\mu\nu}$ is the Kronecker symbol. Thus, formula (14) gives a rigorous sense for $\widetilde{\Delta}^{\{e_n\},1}_L = \int_0^1 a^2_t(dt^2)$ (see [8]). (The formula

$$\widetilde{\Delta}^{\{h_n\},1}_L = \int_0^1\int_0^1 \mathbf{1}_I(s,t)a_s a_t\,(ds\, dt) \tag{15}$$

is probably more accurate. However, there is a conjecture that the formula $\int_0^1 a^2_t(dt^2)$ can be included into the quantum Ito table (see [9]).)

If $\kappa \in E_{\mathbb{C}}^{\otimes 2}$, it is possible to give a rigorous sense for the integral

$$\int_0^1\int_0^1 \kappa_{\mu\nu}(t,s)\dot{a}_t^{\mu}\dot{a}_s^{\nu}\,dt\,ds$$

as $\widetilde{\Xi}_{0,2}\Big(\frac{\partial^2}{\partial t\partial s}\kappa\Big) \in L^b(\mathcal{E}^*,\mathcal{E}^*)$. Let $k_{\mu\nu}=0$ if $\mu\neq\nu$. Let $S\Phi \in C^2_{L,(-1)}(E_{\mathbb{C}},\mathbb{C})$. Then

$$S\Big(\int_0^1\int_0^1 \kappa_{\mu\nu}(t,s)\dot{a}_t^{\mu}\dot{a}_s^{\nu}\,dt\,ds\,\Phi\Big)(\xi) = S\Big(\widetilde{\Xi}_{0,2}\Big(\frac{\partial^2}{\partial t\partial s}\kappa\Big)\Phi\Big)(\xi) = \Big\langle S\Phi''(\xi), \frac{\partial^2}{\partial t\partial s}\kappa\Big\rangle =$$

$$= \sum_{\mu=1}^{d}\Big\langle S\Phi''_{E_\mu E_\mu}(\xi), \frac{\partial^2}{\partial t\partial s}k_{\mu\mu}\Big\rangle = \sum_{\mu=1}^{d}\int_0^1 \kappa_{\mu\mu}(s,t)\nu^{\xi}_{\mu\mu}(ds\,dt). \quad (16)$$

The following theorem is a direct corollary of proposition 2.

Theorem 1. *If* $\Phi \in \operatorname{Dom}\widetilde{\Delta}_L^{\{h_n\},-1}$ *and* $S\Phi \in C^2_{L,(-1)}(E_{\mathbb{C}},\mathbb{C})$. *Then*

$$S(\widetilde{\Delta}_L^{\{h_n\},-1}\Phi)(\xi) = \sum_{\mu=1}^{d}\frac{1}{\pi^2}\int_0^1 \boldsymbol{1}_I(s,t)\nu^{\xi}_{\mu\mu}(ds\,dt). \quad (17)$$

Due to (16), formula (17) gives a rigorous sense for $\widetilde{\Delta}_L^{\{h_n\},-1} = \frac{1}{\pi^2}\int_0^1 \dot{a}_t^2(dt^2)$. (Similarly to (15), the formula

$$\widetilde{\Delta}_L^{\{h_n\},-1} = \frac{1}{\pi^2}\int_0^1\int_0^1 \mathbf{1}_I(s,t)\dot{a}_s\dot{a}_t(ds\,dt)$$

is probably more accurate.)

3. Yang–Mills equations

Let $A(x)=A_\mu(x)dx^\mu$ be a smooth $u(N)$-valued 1-form on $\mathbb{R}^d$. This form determines a connection in the trivial vector bundle with base $\mathbb{R}^d$, fiber $\mathbb{C}^N$, and structure group $U(N)$. The covariant derivative of $C^1(\mathbb{R}^d,u(N))$ is defined by $\nabla_\mu\phi = \partial_\mu\phi + [A_\mu,\phi]$. The curvature $F(x)=\sum_{\mu<\nu}F_{\mu\nu}(x)dx^\mu\wedge dx^\nu$ is determined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The Yang–Mills equations on a connection A have the form

$$\nabla_\mu F^{\mu\nu} = 0. \quad (18)$$

In paper [12] by Accardi, Gibilisco, and Volovich, the following was proved. The parallel transport associated with the connection A is a solution to the Laplace equation for the Lévy Laplacian if and only if A satisfies to the Yang-Mills equations. In [12], the parallel transport was considered as an operator-valued functional on the space of

C^1-smooth curves in $\mathbb{R}^d$ and the Lévy Laplacian was defined as an integral functional determined by the special form of the second derivative. In [14, 15], it was shown that this Laplacian can be defined as the Cesaro mean of the second directional derivatives (this Laplacian coincides with $\pi^2\Delta_L^{\{h_n\},-1}$).

A stochastic parallel transport and its connection to the Lévy Laplacian can be considered. Let $\{b_t\}_{t\in[0,1]}$ be a standard d-dimensional Brownian motion and $(\Omega,\mathcal{F},P)$ be the probability space associated with this process. The stochastic parallel transport $U^A(b,t)$ is a solution to the stochastic equation:

$$U^A(b,t)=I_N-\int_0^t A_\mu(b_s)U^A(b,s)\circ db_s^\mu,$$

where $\circ db$ is the Stratonovich differential.

In paper [13] by Leandre and Volovich, the Lévy Laplacian on the Sobolev space over the Wiener measure P was introduced. This Laplacian was defined as the integral functional. It was shown that the stochastic parallel transport $U^A(b,1)$ is a solution to the Laplace equation for a such Lévy Laplacian if and only if A satisfies to the Yang–Mills equations. In [16], the Lévy Laplacian Δ_L defined as the Cesaro mean of the second directional derivatives on the Sobolev space over the Wiener measure P was introduced. In [16], it was proven that A satisfies to the Yang–Mills equations if and only if $U^A(b,1)$ satisfies

$$\Delta_L U^A(b,1)=U^A(b,1)\int_0^1 U^A(b,t)^{-1}F_{\mu\nu}(b_t)F^{\mu\nu}(b_t)U^A(b,t)\,dt.$$

Thus, in contrast to the deterministic case, the Lévy Laplacian as the integral functional and the Lévy Laplacian as the Cesaro mean are two different operators on the Sobolev space over P.

In [4], it was proven that the Lévy Laplacian Δ_L coincides with $\pi^2\widetilde{\Delta}_L^{\{h_n\},-1}$ under the canonical embedding J of the Sobolev space over the Wiener measure into $M_N(\mathbb{C})\otimes_\pi\mathcal{E}^*$ (the space of $M_N(\mathbb{C})$-valued Hida functionals). Moreover, it is possible to show that S-transform of $JU^A(b,1)$ belongs to $C_L^2(E_\mathbb{C},M_N(\mathbb{C}))$ (this space is defined by analogy with $C_L^2(E_\mathbb{C},\mathbb{C})$). Thus,

$$J(\Delta_L U^A(b,1))=\Big(\int_0^1 \dot{a}_t^2(dt^2)\Big)JU^A(b,1).$$

It would be of interest to investigate whether the definition of the Lévy Laplacian from [13] could be reformulated in terms of the quantum stochastic processes. It is still unknown how the Lévy Laplacians from [13] and [16] are connected.

Conclusions

Further research is needed to see whether the Lévy Laplacian $\Delta_L^{(-1)}$ could be included in the Ito quantum table and used in some areas related to quantum probability (see, e.g., [21–23] and, especially, [9]). In addition, it would be of relevance to study the approach based on the Lévy Laplacian in some areas connected to the theory of gauge fields (see, e.g., [24–27]).

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Volkov Boris Olegovich, Candidate of Physical and Mathematical Sciences, Senior Research Associate; Associate Professor

Steklov Mathematical Institute of Russian Academy of Sciences
ul. Gubkina, 8, Moscow, 119991 Russia

Bauman Moscow State Technical University
ul. Vtoraya Baumanskaya, 5, str. 1, Moscow, 105005 Russia

E-mail: *borisvolkov1986@gmail.com*

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Лапласиан Леви и процесс уничтожения

Б.О. Волков

Математический институт им. В.А. Стеклова Российской академии наук, г. Москва, 119991, Россия
Московский государственный технический университет им. Н.Э. Баумана, г. Москва, 105005, Россия

Аннотация

Лапласианы Леви представляют собой бесконечномерные операторы Лапласа, определенные как среднее Чезаро вторых производных по направлению. В теории распределений Соболева – Шварца над гауссовской мерой на бесконечномерном пространстве (исчислении Хиды) можно рассмотреть два канонических лапласиана Леви. Первый из них, так называемый классический лапласиан Леви, хорошо изучен. Интерес ко второму лапласиану обусловлен его связью с исчислением Маллявэна (теорией пространств Соболева над мерой Винера) и калибровочной теорией Янга – Миллса. Для классического лапласиана Леви известно представление в виде квадратичной функции от процесса уничтожения. Это представление может быть получено с помощью S-преобразования (преобразования

Сигала – Баргмана). В настоящей статье по аналогии показано, что для второго лапласиана Леви существует представление в виде квадратичной функции от производной процесса уничтожения. Полученное представление может оказаться полезным для изучения калибровочных полей и лапласиана Леви в исчислении Маллявэна.

Ключевые слова: Лапласиан Леви, исчисление Хиды, квантовая вероятность, процесс уничтожения

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Волков Борис Олегович, кандидат физико-математических наук, старший научный сотрудник; доцент

Математический институт им. В.А. Стеклова Российской академии наук
ул. Губкина, д. 8, г. Москва, 119991, Россия

Московский государственный технический университет им. Н.Э. Баумана
ул. 2-я Бауманская, д. 5, стр. 1, г. Москва, 105005, Россия

E-mail: *borisvolkov1986@gmail.com*

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