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ON DETECTION OF GAUSSIAN STOCHACTIC VECTORS

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Abstract

The problem of minimax detection of a Gaussian random signal vector in white Gaussian additive noise has been considered. We suppose that an unknown vector $\boldsymbol{\sigma}$ of the signal vector intensities belongs to the given set $\mathcal{E}$. We have investigated when it is possible to replace the set $\mathcal{E}$ (and, in particular, by a single point $\boldsymbol{\sigma}_0$) by a smaller set $\mathcal{E}_0$ without quality loss.

Keywords: detection, minimax, reduction

Introduction

1. Simple hypotheses. There are two simple hypotheses $\mathcal{H}_0$ (“noise”) and $\mathcal{H}_1$ (“noise + stochastic signal”) on observations $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$:

$$\begin{aligned} &\mathcal{H}_0 : \mathbf{y} = \boldsymbol{\xi}, \\ &\mathcal{H}_1 : \mathbf{y} = \mathbf{s} + \boldsymbol{\xi}, \end{aligned} \tag{1}$$

where $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ are the independent $\mathcal{N}(0,1)$-Gaussian random variables, and $\mathbf{s} = (s_1, \dots, s_n)$ are independent on $\boldsymbol{\xi}$, independent $\mathcal{N}(0, \sigma_i^2)$, $i = 1, \dots, n$-Gaussian random variables (i.e., $\mathbf{E}(s_i^2) = \sigma_i^2$). Let us denote $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, where all $\sigma_i \geq 0$, and introduce the function

$$D(\boldsymbol{\sigma}) = \sum_{i=1}^{n} \ln(1 + \sigma_i^2). \tag{2}$$

Then, for conditional probability densities, we have

$$\begin{aligned} p(\mathbf{y}|\mathcal{H}_0) &= (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} y_i^2\right), \\ p(\mathbf{y}|\boldsymbol{\sigma}) &= (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \frac{y_i^2}{(1+\sigma_i^2)} - \frac{1}{2}\, D(\boldsymbol{\sigma})\right). \end{aligned} \tag{3}$$

Let us denote also

$$r(\mathbf{y}, \boldsymbol{\sigma}) = \ln \frac{p(\mathbf{y}|\boldsymbol{\sigma})}{p(\mathbf{y}|\mathcal{H}_0)} = \frac{1}{2}\sum_{i=1}^{n} \frac{\sigma_i^2 y_i^2}{1+\sigma_i^2} - \frac{1}{2}\, D(\boldsymbol{\sigma}). \tag{4}$$

The optimal solution of the problem of testing the simple hypothesis $\mathcal{H}_0$ against the simple alternative $\mathcal{H}_1$ (Neyman–Pearson criteria) has the form

$$\mathbf{y} \in \mathcal{A}(A, \boldsymbol{\sigma}) \Rightarrow \mathcal{H}_0, \quad \mathbf{y} \notin \mathcal{A}(A, \boldsymbol{\sigma}) \Rightarrow \mathcal{H}_1, \tag{5}$$

where the set (ellipsoid)) $\mathcal{A}(A,\boldsymbol{\sigma})$ is

$$\mathcal{A}(A,\boldsymbol{\sigma}) = \left\{ \mathbf{y} : \sum_{i=1}^{n} \frac{\sigma_i^2 y_i^2}{1+\sigma_i^2} \le D(\boldsymbol{\sigma}) + A \right\}, \quad \boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n). \tag{6}$$

The level A of this test is determined by the given first-kind error probability ("*false alarm probability*") $\alpha = \alpha(A,\boldsymbol{\sigma})$:

$$\alpha(A,\boldsymbol{\sigma}) = \mathbf{P}(\mathbf{y} \notin \mathcal{A}|\mathcal{H}_0)) = \mathbf{P}\left(\sum_{i=1}^{n} \frac{\sigma_i^2 \xi_i^2}{1+\sigma_i^2} > D(\boldsymbol{\sigma}) + A \right). \tag{7}$$

If hypothesis $\mathcal{H}_1$ is true, then $y_i = \xi_i + \sigma_i \eta_i \sim \sqrt{1+\sigma_i^2}\,\eta_i$, where $(\eta_1,\dots,\eta_n)$ are independent $\mathcal{N}(0,1)$-Gaussian random variables. The second-kind error probability ("*miss probability*") $\beta(A,\boldsymbol{\sigma})$ is defined by the following formula

$$\beta(A,\boldsymbol{\sigma}) = \mathbf{P}(\mathbf{y} \in \mathcal{A}|\mathcal{H}_1) = \mathbf{P}\left(\sum_{i=1}^{n} \sigma_i^2 \xi_i^2 < D(\boldsymbol{\sigma}) + A \right). \tag{8}$$

For the given value α, we denote by $\beta(\alpha,\boldsymbol{\sigma})$ the minimum possible value $\beta(A,\boldsymbol{\sigma})$ for the optimal choice of the level A (according to formulas (7), (8)).

2. Simple hypothesis against composite alternative. Let a set $\mathcal{E}$ of non-negative vectors $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ be given. Let us assume that on the vector $\boldsymbol{\sigma}$, describing the hypothesis $\mathcal{H}_1$ from (1), it is known only that $\boldsymbol{\sigma} \in \mathcal{E}$, but the vector $\boldsymbol{\sigma}$ itself is not known (i.e., the hypothesis $\mathcal{H}_1$ is composite). Similarly to (5), for testing hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$, we choose a decision region $\mathcal{A} \in \mathbb{R}^n$, such that

$$\mathbf{y} \in \mathcal{A} \Rightarrow \mathcal{H}_0, \qquad \mathbf{y} \notin \mathcal{A} \Rightarrow \mathcal{H}_1.$$

The first- and second-kind error probabilities are defined, respectively, by the following formulas

$$\alpha(\mathcal{A}) = \mathbf{P}(\mathbf{y} \notin \mathcal{A}|\mathcal{H}_0)$$

and

$$\beta(\mathcal{A},\mathcal{E}) = \mathbf{P}(\mathbf{y} \in \mathcal{A}|\mathcal{H}_1) = \sup_{\boldsymbol{\sigma}\in\mathcal{E}} \mathbf{P}(\mathbf{y} \in \mathcal{A}|\boldsymbol{\sigma}).$$

In other words, the minimax problem of testing hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ is considered.

Provided that the first-kind error probability α, $0 < \alpha < 1$, is given, we are interested in finding the minimal possible second-kind error probability

$$\beta(\alpha,\mathcal{E}) = \inf_{\mathcal{A}:\alpha(\mathcal{A})\le\alpha} \beta(\mathcal{A},\mathcal{E}) \tag{9}$$

and the corresponding decision region $\mathcal{A}(\alpha)$.

Without any loss of generality, we assume that the set $\mathcal{E}$ is closed and Lebeques measurable on $\mathbb{R}^n$. Formally speaking, the optimal solution of the problem (9) of minimax testing of hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ is described in Wald's general theory of statistical decisions [1]. For this solution, we need to find the "*least favorable*" prior distribution $\pi_{\rm lf}(d\mathcal{E})$ on $\mathcal{E}$, replace the composite hypothesis $\mathcal{H}_1$ by the simple hypothesis $\mathcal{H}_1(\pi_{\rm lf})$, and then investigate characteristics of the corresponding Neyman-Pearson criteria for testing simple hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1(\pi_{\rm lf})$. Unfortunately, all that can be done only in some very special cases. It is natural then to separate cases when that "least favorable" prior distribution on $\mathcal{E}$ has the simplest form (e.g., it is concentrated in one point from $\mathcal{E}$).

Clearly, for the value $\beta(\alpha, \mathcal{E})$, the lower bound holds

$$\beta(\alpha, \mathcal{E}) \geq \sup_{\boldsymbol{\sigma} \in \mathcal{E}} \beta(\alpha, \boldsymbol{\sigma}). \tag{10}$$

The function $\beta(\alpha, \boldsymbol{\sigma})$, $\alpha \in [0,1]$, $\boldsymbol{\sigma} \in \mathbb{R}_+^n$ is continuous on both arguments. Since the set $\mathcal{E} \in \mathbb{R}_+^n$ is supposed to be closed, then there exists $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_0(\mathcal{E}, \alpha) \in \mathcal{E}$, such that

$$\beta(\alpha, \boldsymbol{\sigma}_0) = \sup_{\boldsymbol{\sigma} \in \mathcal{E}} \beta(\alpha, \boldsymbol{\sigma}).$$

Firstly, we are interested in for what kind of $\mathcal{E}$ the "least favorable" prior distribution is concentrated in the point $\boldsymbol{\sigma}_0$, and then the following equality holds

$$\beta(\alpha, \mathcal{E}) = \beta(\alpha, \boldsymbol{\sigma}_0). \tag{11}$$

If the equality (11) holds for the set $\mathcal{E}$, then, without any loss of detection quality, we may replace the composite hypothesis $\mathcal{H}_1 = \{\mathcal{E}\}$ by the simple hypothesis $\mathcal{H}_1 = \boldsymbol{\sigma}_0$, and the optimal solution (5), (6) for the simple hypothesis $\mathcal{H}_1 = \boldsymbol{\sigma}_0$ remains optimal (in the minimax sense) for the composite hypothesis $\mathcal{H}_1 = \{\mathcal{E}\}$ as well (see similar question for shifts of measures [2]). Some sufficient conditions for having the equality (11) are given below in proposition 2. Clearly, these conditions set rather strong limitations on the set $\mathcal{E}$.

Earlier, it was shown in proposition 1 that sometimes it is possible to replace the set $\mathcal{E}$ by a smaller set $\mathcal{E}_0$ (i.e., to make a reduction of the set $\mathcal{E}$) without any loss of detection quality.

The probability $\beta(\alpha, \mathcal{E})$ should be very small. For this reason, often, instead of the strong condition (11), a simpler asymptotic analogue is investigated comparing exponents of the error probabilities (see, e.g. [3]). In this case, we are interested in validity of a weaker condition:

$$\ln \beta(\alpha, \mathcal{E}) = \ln \beta(\alpha, \boldsymbol{\sigma}) + o\left(\ln \beta(\alpha, \boldsymbol{\sigma})\right), \quad |\ln \beta(\alpha, \boldsymbol{\sigma})| \to \infty. \tag{12}$$

Obviously, the condition (12) should hold under weaker restrictions on the set $\mathcal{E}$ than in the case of the condition (11). These results will be discussed in the forthcoming paper [4].

Below, as usual, $\boldsymbol{\sigma} \leq \boldsymbol{\lambda}$ means $\sigma_i \leq \lambda_i$, $i = 1, \ldots, n$.

1. Results

1. Reduction of the set $\mathcal{E}$. Sometimes it is possible to replace the set $\mathcal{E}$ by a smaller set $\mathcal{E}_0$ without any loss of detection quality. Let us define such set $\mathcal{E}_0 = \mathcal{E}_0(\mathcal{E})$ as any set having the following property:

$$\text{for any} \quad \boldsymbol{\sigma} \in \mathcal{E} \quad \text{there exists} \quad \boldsymbol{\sigma}_0 \in \mathcal{E}_0 \quad \text{with} \quad \boldsymbol{\sigma}_0 \leq \boldsymbol{\sigma}. \tag{13}$$

Since the set $\mathcal{E}$ is closed, then $\mathcal{E}_0 \subseteq \mathcal{E}$. Generally, the set $\mathcal{E}_0$ can be chosen non-uniquely.

We show below that, for any Bayes criteria of testing a simple hypothesis $\mathcal{H}_0$ against a composite alternative $\mathcal{H}_1 = \{\mathcal{E}\}$, the set $\mathcal{E}$ can be replaced by the set $\mathcal{E}_0$ without any loss of quality. It remains valid for likelihood ratio criteria as well. In the one-dimensional case, these properties are similar to the case of distributions with monotone likelyhood ratio [5]. Introduction of such set $\mathcal{E}_0$ (when it is possible) simplifies the test used.

2. Bayes criteria. Let us consider the Bayes criteria with a prior distribution $\pi(d\mathcal{E})$ on $\mathcal{E}$ and corresponding decision set $\mathcal{A} \in \mathbb{R}^n$ ($\mathbf{y} \in \mathcal{A} \Rightarrow \mathcal{H}_0$, $\mathbf{y} \notin \mathcal{A} \Rightarrow \mathcal{H}_1$) of the form

$$\mathcal{A} = \mathcal{A}(A) = \{\mathbf{y} : r(\mathbf{y}, \mathcal{E}, \pi) \leq A\}, \tag{14}$$

where (see (3) and (4))

$$p(\mathbf{y}|\mathcal{H}_1,\pi) = \int\limits_{\boldsymbol{\sigma}\in\mathcal{E}} p(\mathbf{y}|\boldsymbol{\sigma})\pi(d\mathcal{E}) =$$

$$= (2\pi)^{-n/2} \int\limits_{\boldsymbol{\sigma}\in\mathcal{E}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} y_i^2/(1+\sigma_i^2) - \frac{1}{2}\,D(\boldsymbol{\sigma})\right)\pi(d\mathcal{E})$$

and

$$r(\mathbf{y},\mathcal{E},\pi) = \ln\frac{p(\mathbf{y}|\mathcal{H}_1,\pi)}{p(\mathbf{y}|\mathcal{H}_0)} = \ln \int\limits_{\boldsymbol{\sigma}\in\mathcal{E}} \exp\left(\frac{1}{2}\sum_{i=1}^{n} y_i^2\sigma_i^2/(1+\sigma_i^2) - \frac{1}{2}\,D(\boldsymbol{\sigma})\right)\pi(d\mathcal{E}).$$

Then, $\mathcal{A}$ is a convex set in $\mathbb{R}^n$, and if $\mathbf{y} = (y_1,\ldots,y_n) \in \mathcal{A}$, then all $(\pm y_1,\ldots,\pm y_n)$ belong to $\mathcal{A}$, i.e., the set $\mathcal{A}$ is symmetric with respect to any coordinate axis or plane. In particular, such $\mathcal{A}$ is also a centrally symmetric set (i.e., if $\mathbf{y}\in\mathcal{A}$, then $(-\mathbf{y})\in\mathcal{A}$).

Let us assume that the Bayes criteria with a prior distribution $\pi(d\mathcal{E})$ on $\mathcal{E}_0$ is used for $\mathbf{y} = \mathbf{s} + \boldsymbol{\xi}$, $\boldsymbol{\sigma}\in\mathcal{E}$ from (1) and $\mathcal{A}\in\mathbb{R}^n$ of the form (14) is the corresponding decision region. Let us also assume that for the second-kind error probability and some $\beta \ge 0$ we have

$$\beta(\mathcal{A},\boldsymbol{\sigma}_0) = \mathbf{P}(\mathbf{y}\in\mathcal{A}|\boldsymbol{\sigma}_0) = \mathbf{P}\left\{r(\mathbf{y},\mathcal{E}_0,\pi) \le A|\boldsymbol{\sigma}_0\right\} \le \beta, \qquad \boldsymbol{\sigma}_0\in\mathcal{E}_0. \tag{15}$$

Let us show that the inequality (15) remains valid for any $\boldsymbol{\sigma}\in\mathcal{E}$, i.e.,

$$\beta(\mathcal{A},\boldsymbol{\sigma}) = \mathbf{P}(\mathbf{y}\in\mathcal{A}|\boldsymbol{\sigma}) = \mathbf{P}\left\{r(\mathbf{y},\mathcal{E}_0,\pi) \le A|\boldsymbol{\sigma}\right\} \le \beta, \qquad \boldsymbol{\sigma}\in\mathcal{E}. \tag{16}$$

In other words, the second-kind error probability does not increase for any Bayes criteria extension of the set $\mathcal{E}_0$ up to the set $\mathcal{E}$ (the first-kind error probability $\alpha(\mathcal{A})$ does not change). In particular, since $\mathcal{E}_0 \subseteq \mathcal{E}$, we get

$$\beta(\alpha,\mathcal{E}_0) = \beta(\alpha,\mathcal{E}), \qquad 0 \le \alpha \le 1. \tag{17}$$

Let us prove the relation (16). Let $\boldsymbol{\sigma}\in\mathcal{E}$, but $\boldsymbol{\sigma}\notin\mathcal{E}_0$. Then, there exists $\boldsymbol{\sigma}_0\in\mathcal{E}_0$ with $\boldsymbol{\sigma}_0 < \boldsymbol{\sigma}$. Let $\mathbf{s}_0$ be a Gaussian "signal" in (1) in the case of $\boldsymbol{\sigma}_0$. Then, in the case of $\boldsymbol{\sigma}$, such "signal" $\mathbf{s}$ has the form $\mathbf{s} = \mathbf{s}_0 + \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is independent of $\mathbf{s}_0$ Gaussian random vector. The inequality (16) follows from the auxiliary result (the set $\mathcal{A}$ satisfies its conditions).

Lemma 1. *Let $\mathcal{B}\in\mathbb{R}^n$ be a convex set, such that if $\mathbf{y} = (y_1,\ldots,y_n)\in\mathcal{B}$, then all points of the form $(\pm y_1,\ldots,\pm y_n)$ belong to $\mathcal{B}$. Let also $\boldsymbol{\xi},\boldsymbol{\eta}$ be independent zero mean Gaussian vectors consisting of independent (probably, with different distributions) components. Then*

$$\mathbf{P}(\boldsymbol{\xi}+\boldsymbol{\eta}\in\mathcal{B}) \le \mathbf{P}(\boldsymbol{\xi}\in\mathcal{B}). \tag{18}$$

Proof. If $n = 1$, then $\mathcal{B} = [-a,a]$, $a > 0$, and, clearly, the inequality (18) holds. Let $n = 2$ and vectors (ξ_1,ξ_2) and $(\xi_1+\eta_1,\xi_2+\eta_2)$ be compared. We compare first vectors (ξ_1,ξ_2) and $(\xi_1+\eta_1,\xi_2)$. Let us denote

$$\mathcal{B}_x = \{\mathbf{y}\in\mathcal{B} : y_2 = x\} \in \mathbb{R}^1.$$

Due to assumptions of the lemma, we have $\mathcal{B}_x = [-a(x),a(x)]$, $a(x) > 0$ for any x. Therefore, for fixed ξ_2, the problem reduces to the case $n = 1$ and

$$\mathbf{P}\{\xi_1+\eta_1\in\mathcal{B}_{\xi_2}\} \le \mathbf{P}\{\xi_1\in\mathcal{B}_{\xi_2}\}, \tag{19}$$

and then

$$\mathbf{P}\{(\xi_1+\eta_1,\xi_2)\in\mathcal{B}\}\le\mathbf{P}\{(\xi_1,\xi_2)\in\mathcal{B}\}. \tag{20}$$

Let us compare vectors $(\xi_1+\eta_1,\xi_2)$ and $(\xi_1+\eta_1,\xi_2+\eta_2)$. Similarly to (19) and (20), we get

$$\mathbf{P}\{\xi_2+\eta_2\in\mathcal{B}_{\xi_1+\eta_1}\}\le\mathbf{P}\{\xi_2\in\mathcal{B}_{\xi_1+\eta_1}\}$$

and

$$\mathbf{P}\{(\xi_1+\eta_1,\xi_2+\eta_2)\in\mathcal{B}\}\le\mathbf{P}\{(\xi_1+\eta_1,\xi_2)\in\mathcal{B}\}. \tag{21}$$

Then, from (20) and (21), the inequality (18) follows for $n=2$. Similarly, the case $n=3$ reduces to the case $n=2$, etc. It proves the inequality (18) for any n. □

3. Likelihood ratio criteria. For any function $A(\boldsymbol{\sigma})$, the critical region $\mathcal{A}_{ML}(A,\mathcal{E})$ of that criteria is defined by the relation

$$\mathcal{A}_{ML}(A,\mathcal{E})=\left\{\mathbf{y}:\sup_{\boldsymbol{\sigma}\in\mathcal{E}}\left[2r(\mathbf{y},\boldsymbol{\sigma})-A(\boldsymbol{\sigma})\right]\le 0\right\} \tag{22}$$

and then $\mathbf{y}\in\mathcal{A}_{ML}(A,\mathcal{E})\Rightarrow\mathcal{H}_0$, $\mathbf{y}\notin\mathcal{A}_{ML}(A,\mathcal{E})\Rightarrow\mathcal{H}_1$.

We show that, without any loss of quality, we can replace the set $\mathcal{E}$ in (22) by a smaller set $\mathcal{E}_0$ (see (13)), i.e., we can use the criteria:

$$\mathcal{A}_{MLR}(A,\mathcal{E})=\left\{\mathbf{y}:\sup_{\boldsymbol{\sigma}\in\mathcal{E}_0}\left[2r(\mathbf{y},\boldsymbol{\sigma})-A(\boldsymbol{\sigma})\right]\le 0\right\}, \tag{23}$$

keeping the same decision making method. In other words, the second-kind error probability does not increase (the 1-st kind error probability $\alpha(\mathcal{A})$ does not change) for likelihood ratio criteria expansion of the set $\mathcal{E}_0$ up to the set $\mathcal{E}$.

Indeed, if $\boldsymbol{\sigma}\in\mathcal{E}$, but $\boldsymbol{\sigma}\notin\mathcal{E}_0$, then there exists $\boldsymbol{\sigma}_0\in\mathcal{E}_0$ with $\boldsymbol{\sigma}_0<\boldsymbol{\sigma}$. Using the definition (23) and formulas (25) and (26) below, we have

$$\begin{aligned}\beta(A,\boldsymbol{\sigma})&=\mathbf{P}\left\{\sup_{\boldsymbol{\lambda}\in\mathcal{E}_0}\left[2r(\mathbf{y},\boldsymbol{\lambda})-A(\boldsymbol{\lambda})\right]\le 0|\boldsymbol{\sigma}\right\}=\\&=\mathbf{P}\left\{\sup_{\boldsymbol{\lambda}\in\mathcal{E}_0}\left[\sum_{i=1}^{n}\frac{\lambda_i^2(1+\sigma_i^2)\eta_i^2}{1+\lambda_i^2}-D(\boldsymbol{\lambda})-A(\boldsymbol{\lambda})\right]\le 0\right\}\le\\&\le\mathbf{P}\left\{\sup_{\boldsymbol{\lambda}\in\mathcal{E}_0}\left[\sum_{i=1}^{n}\frac{\lambda_i^2(1+\sigma_{0i}^2)\eta_i^2}{1+\lambda_i^2}-D(\boldsymbol{\lambda})-A(\boldsymbol{\lambda})\right]\le 0\right\}=\\&=\mathbf{P}\left\{\sup_{\boldsymbol{\lambda}\in\mathcal{E}_0}\left[2r(\mathbf{y},\boldsymbol{\lambda})-A(\boldsymbol{\lambda})\right]\le 0|\boldsymbol{\sigma}_0\right\}=\beta(A,\boldsymbol{\sigma}_0).\end{aligned} \tag{24}$$

The obtained results (16) and (24) can be formulated as follows.

Proposition 1. *Let us consider the minimax problem of testing a simple hypothesis $\mathcal{H}_0$ against a composite alternative $\mathcal{H}_1=\{\mathcal{E}_0\}$ and let $\mathcal{E}_0\subseteq\mathcal{E}$. If for the set $\mathcal{E}$ the condition* (13) *is satisfied, then, for any Bayes criteria and the likelihood ratio criteria, the first- and second-kind error probabilities do not change if the set $\mathcal{E}_0$ is replaced by the set $\mathcal{E}$. In particular, the equality* (17) *holds.*

Remark 1. In proposition 1, it would be more natural to start with a set $\mathcal{E}$ and replace it by a set $\mathcal{E}_0\subseteq\mathcal{E}$. However, in this case, it would be necessary to describe “projections” of Bayes criteria from $\mathcal{E}$ on $\mathcal{E}_0$.

Remark 2. Similarly to $\mathcal{E}_0$,“reduced” sets $\mathrm{red}_1 S$ and $\mathrm{red}_2 S$ have been introduced earlier in [2], where Gaussian measures had only different shifts. From the analytical viewpoint, various convexity properties with respect to shifts of the Gaussian measures were very useful in [2]. For example, owing to them, the set $\mathrm{red}_1 S$ had a very simple and natural form. Unfortunately, the author does not know similar convexity properties concerning variances of the Gaussian measures and, for this reason, only certain monotonicity properties have been used (which is less productive).

4. Exact equality (11)**.** The formula (11) has also another equivalent interpretation. Let us assume that we know it initially that the “signal” in the hypothesis $\mathcal{H}_1$ is a certain $\boldsymbol{\sigma}$, and, therefore, we use the optimal solution (5), (6) for that $\boldsymbol{\sigma}$. We assume additionally that, in fact, the “signal” in the hypothesis $\mathcal{H}_1$ can also take other values $\boldsymbol{\lambda}$ from a set $\mathcal{E}$. For what $\mathcal{E}$ does the solution (5)–(6) (oriented only on $\boldsymbol{\sigma}$) remain optimal for the set $\mathcal{E}$ as well?

If $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\lambda}$ and decision (5), (6) is used, then the first-kind error probability α does not change. Therefore, it is necessary to check only how the second-kind error probability $\beta_{\boldsymbol{\sigma}}(A,\boldsymbol{\lambda})$ may change

$$\beta_{\boldsymbol{\sigma}}(A,\boldsymbol{\lambda}) = \mathbf{P}(\mathbf{y}\in\mathcal{A}|\boldsymbol{\lambda}) = \mathbf{P}\left(\sum_{i=1}^{n}\frac{\sigma_i^2(\xi_i+s_i)^2}{1+\sigma_i^2} - D(\boldsymbol{\sigma}) < A|\boldsymbol{\lambda}\right) = \\ = \mathbf{P}\left(\sum_{i=1}^{n}\nu_i^2\xi_i^2 - D(\boldsymbol{\sigma}) < A\right), \quad (25)$$

since $(\xi_i+s_i)^2 = (1+\lambda_i^2)\eta_i^2$, $i=1,\ldots,n$, and where

$$\nu_i^2 = \frac{\sigma_i^2(1+\lambda_i^2)}{1+\sigma_i^2} = \sigma_i^2 + \frac{\sigma_i^2(\lambda_i^2-\sigma_i^2)}{1+\sigma_i^2}, \quad i=1,\ldots,n, \quad (26)$$

and $\{\eta_i\}$ are independent $\mathcal{N}(0,1)$-Gaussian random variables.

If, for any $\boldsymbol{\lambda}\in\mathcal{E}$ and A, the following inequality holds ($\boldsymbol{\nu}=(\nu_1,\ldots,\nu_n)$ is defined in (26))

$$\beta_{\boldsymbol{\sigma}}(A,\boldsymbol{\lambda}) = \mathbf{P}\left(\sum_{i=1}^{n}\nu_i^2\xi_i^2 - D(\boldsymbol{\sigma}) < A\right) \le \\ \le \mathbf{P}\left(\sum_{i=1}^{n}\sigma_i^2\xi_i^2 - D(\boldsymbol{\sigma}) < A\right) = \beta(A,\boldsymbol{\sigma}), \quad (27)$$

then

$$\beta(A,\mathcal{E}) \le \sup_{\boldsymbol{\lambda}\in\mathcal{E}}\beta_{\boldsymbol{\sigma}}(A,\boldsymbol{\lambda}) \le \beta(A,\boldsymbol{\sigma})$$

and, therefore, the formula (11) is valid.

Some results showing validity of the inequality (27) for certain $\boldsymbol{\sigma},\boldsymbol{\nu},A$ can be found, for example, in [6–8].

Comparing (8), (25) and (26), we get simple

Proposition 2. 1) *If $\boldsymbol{\sigma}\le\boldsymbol{\lambda}$, then $\beta(A,\boldsymbol{\lambda})\le\beta(A,\boldsymbol{\sigma})$ and $\beta_{\boldsymbol{\sigma}}(A,\boldsymbol{\lambda})\le\beta(A,\boldsymbol{\sigma})$ for any A.*

2) *If $\boldsymbol{\sigma}\le\boldsymbol{\lambda}$ for any $\boldsymbol{\lambda}\in\mathcal{E}$, then $\beta(\alpha,\mathcal{E})=\beta(\alpha,\boldsymbol{\sigma})$ for any α.*

Let us consider the following result as an example, which is the part of lemma 1 from [8].

Lemma 2. *Let us assume that the set of indices $I = \{1, 2, \ldots, n\}$ of vectors $\boldsymbol{\sigma}$, $\boldsymbol{\lambda}$ can be partitioned in $k \geq 1$ groups $I_1, \ldots, I_k$, such that $I = \bigcup_{j=1}^{k} I_j$, $I_i \bigcap I_j = \emptyset$, $i \neq j$, and the following conditions are fulfilled*

$$\sigma_i \leq \lambda_{0,j}, \qquad i \in I_j, \quad j = 1, \ldots, k,$$

where

$$\lambda_{0,j} = \left(\prod_{i \in I_j} \lambda_i \right)^{1/|I_j|}.$$

Then, $\beta(A, \boldsymbol{\lambda}) \leq \beta(A, \boldsymbol{\sigma})$ for any A.

Example 1. Let for given $D > 0$

$$\mathcal{E} = \left\{ \boldsymbol{\lambda} \geq \mathbf{0} : \prod_{i=1}^{n} \left(1 + \lambda_i^2\right) \geq \left(1 + D^2\right)^n \right\}.$$

Then, by formula (26) and lemma 2 with $k = 1$, it follows that the set $\mathcal{E}$ can be replaced (without any loss of quality) by one point $\boldsymbol{\sigma}_0 = (D, \ldots, D) \in \mathcal{E}$ (in the sense of equality (11)).

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References

1. Wald A. *Statistical Decision Functions.* New York, John Wiley & Sons, 1950. ix+179 p.

2. Burnashev M.V. On the minimax detection of an inaccurately known signal in a white Gaussian noise background. *Theory Probab. Appl.*, 1979, vol. 24, no. 1, pp. 106–119. doi: 10.1137/1124008.

3. Zhang W., Poor H.V. On minimax robust detection of stationary Gaussian signals in white Gaussian noise. *IEEE Trans. Inf. Theory*, 2011, vol. 57, no. 6, pp. 3915–3924. doi: 10.1109/TIT.2011.2136210.

4. Burnashev M.V. On detection of Gaussian stochastic sequences. *Probl. Inf. Transm.*, 2017, vol. 53, no. 4. pp. 349–367. doi: 10.1134/S0032946017040044.

5. Lehmann E.L. *Testing of Statistical Hypotheses.* New York, John Wiley & Sons, 1959. 369 p.

6. Ponomarenko L.S. On estimating distributions of normalized quadratic forms of normally distributed random variables. *Theory Probab. Appl.*, 1985, vol. 30, no. 3, pp. 580–584. doi: 10.1137/1130071.

7. Bakirov N.K. Comparison theorems for distribution functions of quadratic forms of Gaussian vectors. *Theory Probab. Appl.*, 1995, vol. 40, no. 2, pp. 340–348. doi: 10.1137/1140035.

8. Burnashev M.V. Two theorems on distribution of gaussian quadratic forms. *Probl. Inf. Transm.*, 2017, vol. 53, no. 3, pp. 3–15.

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Об обнаружении гауссовских сигналов в гауссовском белом шуме

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Аннотация

Изучается проблема минимаксного обнаружения случайного гауссовского векторного сигнала в аддитивном гауссовском белом шуме. Предполагается, что неизвестный вектор $\boldsymbol{\sigma}$ интенсивностей вектора сигналов принадлежит заранее заданному множеству $\mathcal{E}$. Исследуются случаи, когда возможно заменить множество $\mathcal{E}$ на меньшее множество $\mathcal{E}_0$ без потерь в качестве правила обнаружения. В частности, исследуется возможность замены единственной точкой $\boldsymbol{\sigma}_0$.

Ключевые слова: обнаружение сигнала, минимакс, сокращение

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